SOLUTIONS TO TOPIC 1 (ALGEBRA)

1 **a** $18+16+14+12+....$ is an arithmetic series with $u_1 = 18$, $d = -2$ and $n = 30$.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore\ S_{30} = \tfrac{30}{2}(2\times 18 + 29 \times (-2))$$
$$= -330$$

a $48+24+12+6+....$ is a geometric series with $u_1 = 48$, $r = \frac{1}{2}$ and $n = 30$.

$$S_n = \frac{u_1(1-r^n)}{1-r}$$
$$\therefore\ S_{30} = \frac{48\left(1-(\frac{1}{2})^{30}\right)}{1-\frac{1}{2}}$$
$$\approx 96.0$$

2
$$u_1 = 27 \text{ and } u_4 = 8$$
$$\therefore\ u_1 r^3 = 8$$
$$\therefore\ r^3 = \tfrac{8}{27}$$
$$\therefore\ r = \tfrac{2}{3}$$
$$\therefore \text{ the series has sum } S = \frac{u_1}{1-r}$$
$$= \frac{27}{1-\frac{2}{3}}$$
$$= 81$$

3 **a**
$$u_n = \frac{2n+1}{3}$$
$$\therefore\ u_{n+1} - u_n = \frac{2(n+1)+1}{3} - \frac{2n+1}{3}$$
$$= \frac{2n+3-2n-1}{3}$$
$$= \tfrac{2}{3} \text{ for all } n \in \mathbb{Z}^+$$

$\therefore$ consecutive terms differ by $\frac{2}{3}$, so the sequence is arithmetic with $d = \frac{2}{3}$.

b $u_1 = \frac{2(1)+1}{3} = 1$ and $d = \frac{2}{3}$

Now $u_n = u_1 + (n-1)d$

$$\therefore\ u_{50} = 1 + 49 \times \tfrac{2}{3}$$
$$= 33\tfrac{2}{3}$$
$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore\ S_{50} = \tfrac{50}{2}(1 + 33\tfrac{2}{3})$$
$$= 866\tfrac{2}{3}$$

c
$$\text{Let } u_n = 117$$
$$\therefore\ u_1 + (n-1)d = 117$$
$$\therefore\ 1 + (n-1)\tfrac{2}{3} = 117$$
$$\therefore\ n - 1 = 174$$
$$\therefore\ n = 175$$

So, 117 *is* a term of the sequence (it is the 175th term).

d **i** $\sum_{n=1}^{40} u_n = S_{40}$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
$$\therefore\ S_{40} = \tfrac{40}{2}(2\times 1 + 39 \times \tfrac{2}{3})$$
$$= 560$$

ii $\sum_{n=30}^{60} u_n$

$$= S_{60} - S_{29}$$
$$= \tfrac{60}{2}\left(2(1) + 59(\tfrac{2}{3})\right) - \tfrac{29}{2}\left(2(1) + 28(\tfrac{2}{3})\right)$$
$$= 1240 - 299\tfrac{2}{3}$$
$$= 940\tfrac{1}{3}$$

4 **a**
$$u_7 = 1 \quad\text{and}\quad u_{15} = -23$$
$$\therefore\ u_1 + 6d = 1 \quad\text{and}\quad u_1 + 14d = -23$$
$$\therefore\ u_{15} - u_7 = (u_1 + 14d) - (u_1 + 6d)$$
$$= -23 - 1$$
$$\therefore\ 8d = -24$$
$$\therefore\ d = -3$$
$$\text{and } u_1 = 1 - 6d$$
$$= 19$$
$$\text{So, } u_{27} = u_1 + 26d$$
$$= 19 + 26(-3)$$
$$= -59$$

b
$$S_n = \frac{n}{2}(u_1 + u_n)$$
$$\therefore\ S_{27} = \tfrac{27}{2}(u_1 + u_{27})$$
$$= \tfrac{27}{2}(19 - 59)$$
$$= -540$$

5 **a**
$$\frac{u_1}{1-r} = 1.5$$
$$\text{and } u_1 = 1$$
$$\therefore\ 1 - r = \tfrac{1}{1.5}$$
$$\therefore\ 1 - r = \tfrac{2}{3}$$
$$\therefore\ r = \tfrac{1}{3}$$

b
$$S_n = \frac{u_1(1-r^n)}{1-r}$$
$$\therefore\ S_7 = \frac{u_1(1-r^7)}{1-r}$$
$$= \frac{1\left(1-(\frac{1}{3})^7\right)}{1-\frac{1}{3}}$$
$$= \tfrac{3}{2}\left(1 - \tfrac{1}{2187}\right)$$
$$= \tfrac{1093}{729}$$

6 **a** Since they are consecutive terms of an arithmetic sequence,

$$(k^2+5) - (3k) = 3k - (k+1)$$
$$\therefore\ k^2 - 3k + 5 = 2k - 1$$
$$\therefore\ k^2 - 5k + 6 = 0$$
$$\therefore\ (k-2)(k-3) = 0$$
$$\therefore\ k = 2 \text{ or } 3$$

b Since they are consecutive terms of a geometric sequence,

$$\frac{k^2+5}{3k} = \frac{3k}{k+1} \quad \{\text{equating ratios}\}$$
$$\therefore\ k^3 + k^2 + 5k + 5 = 9k^2$$
$$\therefore\ k^3 - 8k^2 + 5k + 5 = 0$$

Using technology, $k \approx 1.32$ is the only solution which satisfies $0 < k < 5$.

7 **a**
$$\frac{u_{n+1}}{u_n} = \frac{12\left(\frac{2}{3}\right)^n}{12\left(\frac{2}{3}\right)^{n-1}}$$
$$= \tfrac{2}{3} \text{ for all } n \in \mathbb{Z}^+$$

So, consecutive terms have a common ratio of $\frac{2}{3}$.

Thus, the sequence is geometric with $r = \frac{2}{3}$.

b
$$u_5 = 12\left(\tfrac{2}{3}\right)^4$$
$$= 12\left(\tfrac{16}{81}\right)$$
$$= \tfrac{64}{27}$$

c **i** $\sum_{n=1}^{\infty} u_n = \frac{u_1}{1-r}$
$= \frac{12\left(\frac{2}{3}\right)^0}{1-\frac{2}{3}}$
$= \frac{12}{\frac{1}{3}}$
$= 36$

ii $\sum_{n=1}^{20} u_n = S_{20}$
$S_n = \frac{u_1(1-r^n)}{1-r}$
$\therefore \quad S_{20} = \frac{12\left(1-(\frac{2}{3})^{20}\right)}{1-\frac{2}{3}}$
≈ 35.9892

8 **a** $u_3 = 20$ and $u_6 = 160$

$\therefore \quad u_1r^2 = 20$ and $u_1r^5 = 160$

$\therefore \quad \frac{u_1r^5}{u_1r^2} = \frac{160}{20}$

$\therefore \quad r^3 = 8$

$\therefore \quad r = 2$

and $u_1 2^2 = 20$

$\therefore \quad u_1 = 5$

b $u_{10} = 5 \times 2^9$
$= 2560$

$\sum_{n=1}^{12} u_n = S_{12}$

Now $S_n = \frac{u_1(r^n-1)}{r-1}$

$\therefore \quad S_{12} = \frac{5(2^{12}-1)}{2-1}$
$= 20\,475$

9 Let u_1, u_2 and u_3 be consecutive terms of an arithmetic sequence with difference d.

$\therefore \quad u_1 = u_2 - d$ and $u_3 = u_2 + d$.

Now $u_1 + u_2 + u_3 = 18$

$\therefore \quad u_2 - d + u_2 + u_2 + d = 18$

$\therefore \quad 3u_2 = 18$

$\therefore \quad u_2 = 6.$

Also, $u_1^2 + u_2^2 + u_3^2 = 396$

$\therefore \quad (6-d)^2 + 6^2 + (6+d)^2 = 396$

$\therefore \quad 36 - 12d + d^2 + 36 + 36 + 12d + d^2 = 396$

$\therefore \quad 2d^2 = 288$

$\therefore \quad d^2 = 144$

$\therefore \quad d = \pm 12$

So, the numbers are -6, 6 and 18.

10 $u_1 = 18$ and $d = -3$.
If the series has n terms, then

$S_n = -210$

$\therefore \quad \frac{n}{2}(2u_1 + (n-1)d) = -210$

$\therefore \quad \frac{n}{2}\left(2 \times 18 + (n-1) \times (-3)\right) = -210$

$\therefore \quad \frac{n}{2}(36 - 3n + 3) = -210$

$n(39 - 3n) = -420$

$\therefore \quad 3n^2 - 39n - 420 = 0$

$\therefore \quad 3(n^2 - 13n - 140) = 0$

$\therefore \quad 3(n-20)(n+7) = 0$

$\therefore \quad n = 20 \quad \{n > 0\}$

So there are 20 terms in the series.

11 Time period = 33 months = 11 quarters
Interest rate = 8% p.a. = 2% per quarter

$\therefore \quad r = 1.02$

$\therefore$ the amount after 11 quarters is

$u_{12} = u_1 \times r^{11}$
$= 3500 \times 1.02^{11}$
≈ 4351.8101

So, the maturing value is £4351.81.

12 The sum of the integers between 100 and 200 which are *not* a multiple of 4, is $(100 + 101 + 102 + \ldots\ldots + 199 + 200)$
$-(100 + 104 + \ldots\ldots + 196 + 200)$.

Now $100 + 101 + 102 + \ldots\ldots + 199 + 200$ is an arithmetic series with $u_1 = 100$, $n = 101$ and $u_n = 200$.

$S_n = \frac{n}{2}(u_1 + u_n)$

$\therefore \quad S_{101} = \frac{101}{2}(100 + 200)$
$= 15\,150$

Now $100 + 104 + \ldots\ldots + 196 + 200$ in an arithmetic series with $u_1 = 100$, $n = 26$ and $u_n = 200$

$\therefore \quad S_{26} = \frac{26}{2}(100 + 200)$
$= 3900$

$\therefore$ the required sum $= 15\,150 - 3900$
$= 11\,250$

13 **a** Let $u_k = 3k - 11$

$\therefore \quad u_1 = 3 \times 1 - 11$
$= -8$

$u_{k+1} - u_k = (3(k+1) - 11) - (3k - 11)$
$= 3k + 3 - 11 - 3k + 11$
$= 3$

Since the difference between consecutive terms is constant, u_k is an arithmetic sequence with $u_1 = -8$ and $d = 3$.

$\therefore \quad \sum_{k=1}^{n} (3k - 11) = S_n$

$\therefore \quad 5536 = \frac{n}{2}(2u_1 + (n-1)d)$

$\therefore \quad 5536 = \frac{n}{2}(2 \times (-8) + (n-1) \times 3)$

$\therefore \quad 11\,072 = n(-16 + 3n - 3)$

$\therefore \quad 3n^2 - 19n - 11\,072 = 0$

$\therefore \quad (3n + 173)(n - 64) = 0$

$n = 64$ or $-\frac{173}{3}$

Since n must be an integer, $n = 64$.

b Let $u_k = \left(\frac{y}{5}\right)^{k-1}$

$\therefore \quad u_1 = \left(\frac{y}{5}\right)^{1-1}$
$= \left(\frac{y}{5}\right)^0$
$= 1$

and $\frac{u_{k+1}}{u_k} = \frac{\left(\frac{y}{5}\right)^k}{\left(\frac{y}{5}\right)^{k-1}}$
$= \frac{y}{5}$ for all $k \in \mathbb{Z}^+$

Since the ratio between consecutive terms is constant, u_k is a geometric sequence with $u_1 = 1$ and $r = \frac{y}{5}$.

$\sum_{k=1}^{\infty} \left(\frac{y}{5}\right)^{k-1} = S = \frac{u_1}{1-r}$

$\therefore \quad 5 = \frac{1}{1-\frac{y}{5}}$

$\therefore \quad \left(1 - \frac{y}{5}\right) \times 5 = 1$

$\therefore \quad 5 - y = 1$

$\therefore \quad y = 4$

14 $\dfrac{x^a\sqrt{x^{3a}}}{x^{-2a}}$
$= x^a(x^{3a})^{\frac{1}{2}} \times x^{-(-2a)}$
$= x^a \times x^{\frac{3a}{2}} \times x^{2a}$
$= x^{\frac{9a}{2}}$

15 $\left(\dfrac{3x^{-1}}{2a^2}\right)^{-2} \times \left(\dfrac{4x^2}{27a^{-3}}\right)^{-1}$
$= \left(\dfrac{2a^2}{3x^{-1}}\right)^2 \times \left(\dfrac{27a^{-3}}{4x^2}\right)$
$= \left(\dfrac{4a^4}{9x^{-2}}\right) \times \left(\dfrac{27a^{-3}}{4x^2}\right)$
$= \dfrac{108a}{36}$
$= 3a$

16 $8^{2x-3} = 16^{2-x}$
$\therefore\ (2^3)^{2x-3} = (2^4)^{2-x}$
$\therefore\ 2^{6x-9} = 2^{8-4x}$
$\therefore\ 6x - 9 = 8 - 4x$
$\therefore\ 10x = 17$
$\therefore\ x = \frac{17}{10}$

17 $\dfrac{3^{x+1} - 3^x}{2(3^x) - 3^{x-1}}$
$= \dfrac{3^{x-1}(3^2 - 3)}{3^{x-1}(2 \times 3 - 1)}$
$= \frac{6}{5}$

18 $2^{2x} - 17(2^{x-1}) + 4 = 0$
$\therefore\ 2 \times 2^{2x} - 17(2^x) + 8 = 0$
$\therefore\ 2X^2 - 17X + 8 = 0\ \ \{X = 2^x\}$
$\therefore\ (2X-1)(X-8) = 0$
$\therefore\ X = \frac{1}{2}$ or 8
$\therefore\ 2^x = 2^{-1}$ or 2^3
$\therefore\ x = -1$ or 3

19 $2^a 8^b = \frac{1}{2}$ and $\dfrac{3^{-a}}{3^{b+1}} = 9$
$\therefore\ 2^a(2^3)^b = 2^{-1}$ and $3^{-a}3^{-(b+1)} = 3^2$
$\therefore\ 2^{a+3b} = 2^{-1}$ and $3^{-a-b-1} = 3^2$
So $a + 3b = -1$ ...... (1)
$-a - b - 1 = 2$ ...... (2)
From (2): $-a - b = 3$
$\therefore\ a = -3 - b$
Substitute into (1): $-3 - b + 3b = -1$
$\therefore\ 2b = 2$
$\therefore\ b = 1$
$\therefore\ a = -4$

20 **a** $\log_4 8 = \dfrac{\log 8}{\log 4}$
$= \dfrac{3\log 2}{2\log 2}$
$= \frac{3}{2}$

b $\log_9\left(\frac{1}{27}\right) = \dfrac{\log\left(\frac{1}{27}\right)}{\log 9}$
$= \dfrac{-3\log 3}{2\log 3}$
$= -\frac{3}{2}$

c $\log_{\frac{1}{3}}\sqrt{3} = \dfrac{\log\sqrt{3}}{\log\frac{1}{3}}$
$= \dfrac{\frac{1}{2}\log 3}{-\log 3}$
$= -\frac{1}{2}$

21 $\log_5(2x-1) = -1$
$\therefore\ 5^{-1} = 2x - 1$
$\therefore\ \frac{1}{5} = 2x - 1$
$\therefore\ \frac{6}{5} = 2x$
$\therefore\ x = \frac{3}{5}$

22 $\log_b A = \dfrac{\log_c A}{\log_c b}$
$\therefore\ \log_5 9 = \dfrac{\log_3 9}{\log_3 5}$

$\therefore\ \dfrac{8}{\log_5 9} = \dfrac{8}{\frac{\log_3 9}{\log_3 5}}$
$= \dfrac{8\log_3 5}{\log_3 9}$
$= \dfrac{8\log_3 5}{2}$
$= 4\log_3 5$

23 $2\ln x + \ln(x-1) - \ln(x-2)$
$= \ln x^2 + \ln(x-1) - \ln(x-2)$
$= \ln\left(\dfrac{x^2(x-1)}{(x-2)}\right)$

24 $\log_3 x + \log_3(x-2) = 1$
$\therefore\ \log_3(x(x-2)) = 1$
$\therefore\ 3^1 = x(x-2)$
$\therefore\ 3 = x^2 - 2x$
$\therefore\ x^2 - 2x - 3 = 0$
$\therefore\ (x-3)(x+1) = 0$
$\therefore\ x = 3\ \ \{x > 2\}$

25 **a** $\log_a(5a)$
$= \log_a 5 + \log_a a$
$= x + 1$

b $\log_a\left(\dfrac{a^2}{25}\right)$
$= \log_a a^2 - \log_a 25$
$= 2 - 2\log_a 5$
$= 2 - 2x$

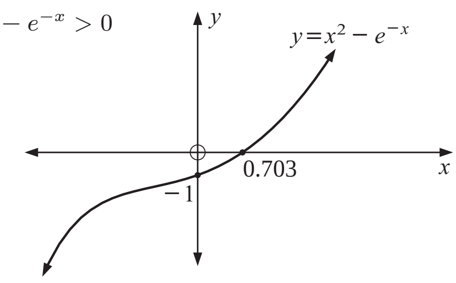
26 **a** $M = ab^3$
$\therefore\ \log_b M = \log_b(ab^3)$
$\therefore\ \log_b M = \log_b a + \log_b b^3$
$\therefore\ \log_b M = \log_b a + 3$

b $D = \dfrac{a}{b^2}$
$\therefore\ \log_b D = \log_b\left(\dfrac{a}{b^2}\right)$
$\therefore\ \log_b D = \log_b a - \log_b b^2$
$\therefore\ \log_b D = \log_b a - 2$

27 **a** $\log_{10} M = 2x - 1$
$\therefore\ 10^{\log_{10} M} = 10^{2x-1}$
$\therefore\ M = 10^{2x-1}$

b $\log_a N = 2\log_a d - \log_a c$
$\therefore\ \log_a N = \log_a d^2 - \log_a c$
$\therefore\ \log_a N = \log_a\left(\dfrac{d^2}{c}\right)$
$\therefore\ N = \dfrac{d^2}{c}$

28 $x^2 > e^{-x} \Leftrightarrow x^2 - e^{-x} > 0$



Using technology the graph meets the x-axis at $x \approx 0.703$
Hence $x^2 > e^{-x}$ if $x > 0.703$

29 $2^{x-1} = 3^{2-x}$

Take the logarithm to base 10 to get:

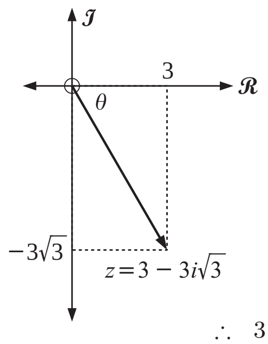
$$\log(2^{x-1}) = \log(3^{2-x})$$
$$\therefore\ (x-1)\log 2 = (2-x)\log 3$$
$$\therefore\ x\log 2 - \log 2 = 2\log 3 - x\log 3$$
$$\therefore\ x\log 2 + x\log 3 = 2\log 3 + \log 2$$
$$\therefore\ x(\log 2 + \log 3) = \log 3^2 + \log 2$$
$$\therefore\ x\log 6 = \log 18$$
$$\therefore\ x = \frac{\log 18}{\log 6}$$
$$\therefore\ x = \log_6 18$$

So, $a = 6$ and $b = 18$.

30 $(2-ai)^3 = (2-ai)^2(2-ai)$

$$= (4 - 2\times 2ai + (ai)^2)(2-ai)$$
$$= (4 - 4ai - a^2)(2-ai)$$
$$= 8 - 4ai - 8ai + 4a^2i^2 - 2a^2 + a^3 i$$
$$= 8 - 12ai - 4a^2 - 2a^2 + a^3 i$$
$$= (8 - 6a^2) + i(a^3 - 12a)$$

31 Let $z = 3 - 3i\sqrt{3}$



$$|z| = \sqrt{3^2 + (3\sqrt{3})^2}$$
$$= \sqrt{9+27} = 6$$
$$\cos\theta = \tfrac{3}{6} = \tfrac{1}{2}$$
$$\therefore\ \theta = \tfrac{\pi}{3}, \text{ and so } \arg(z) = -\tfrac{\pi}{3}$$
$$\therefore\ 3 - 3i\sqrt{3} = 6\operatorname{cis}\left(-\tfrac{\pi}{3}\right)$$
$$= 6\left(\cos(-\tfrac{\pi}{3}) + i\sin(-\tfrac{\pi}{3})\right)$$

32
$$\left(\cos(\tfrac{2\pi}{3}) - i\sin(\tfrac{2\pi}{3})\right)^{10}$$
$$= \left(\cos(-\tfrac{2\pi}{3}) + i\sin(-\tfrac{2\pi}{3})\right)^{10}$$
$$= \left(\operatorname{cis}(-\tfrac{2\pi}{3})\right)^{10}$$
$$= \operatorname{cis}\left(-\tfrac{20\pi}{3}\right)$$
$$= \cos\left(-\tfrac{20\pi}{3}\right) + i\sin\left(-\tfrac{20\pi}{3}\right)$$
$$= \cos\left(-\tfrac{2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right)$$
$$= -\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}$$

33 **a**
$$\frac{3+4i}{1-3i}\times\left(\frac{1+3i}{1+3i}\right)$$
$$= \frac{(3+4i)(1+3i)}{(1-3i)(1+3i)}$$
$$= \frac{3+9i+4i+12i^2}{1-9i^2}$$
$$= \frac{3+13i-12}{1-9(-1)}$$
$$= \frac{-9+13i}{10}$$
$$= \tfrac{-9}{10} + \tfrac{13}{10}i$$

b
$$\frac{3}{i}\left(\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}\right)^2$$
$$= \frac{3}{i}\left(\frac{1-2i}{\sqrt{5}}\right)^2$$
$$= \frac{3}{i}\left(\frac{(1-2i)^2}{5}\right)$$
$$= \frac{3}{i}\left(\frac{1-4i+4i^2}{5}\right)$$
$$= \frac{3}{i}\left(\frac{-3-4i}{5}\right)$$
$$= \frac{-9-12i}{5i}$$
$$= \frac{-9i-12i^2}{5i^2}$$
$$= \frac{-9i+12}{-5}$$
$$= -\tfrac{12}{5} + \tfrac{9}{5}i$$

34
$$\frac{z+2}{z-2} = i$$
$$\therefore\ z+2 = i(z-2)$$
Let $z = a+bi$
$$\therefore\ (a+bi)+2 = i\left((a+bi)-2\right)$$
$$\therefore\ (2+a)+bi = ai + bi^2 - 2i$$
$$\therefore\ (2+a)+bi = -b + (a-2)i$$

Equating real and imaginary parts:

$$2 + a = -b \quad \text{...... (1)}$$
$$\text{and } b = a - 2 \quad \text{...... (2)}$$

Substituting (2) into (1): $2 + a = 2 - a$

$$\therefore\ 2a = 0$$
$$\therefore\ a = 0$$
$$\therefore\ b = -2$$

So, $z = -2i$.

35 Let $z = a+bi$, so $z^* = a - bi$, $a, b \in \mathbb{R}$.

Now, $z^2 = (z^*)^2$

So $(a+bi)^2 = (a-bi)^2$

$$a^2 + 2abi + b^2i^2 = a^2 - 2abi + b^2i^2$$
$$(a^2-b^2) + 2abi = (a^2-b^2) - 2abi$$

Equating imaginary parts gives

$$2ab = -2ab$$
$$\therefore\ 4ab = 0$$
$$\therefore\ a = 0 \text{ or } b = 0.$$

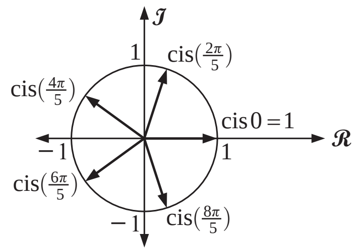
So $z = a$ or $z = bi$, $a, b \in \mathbb{R}$.

So z is either real or purely imaginary.

36 **a** $1 = \operatorname{cis} 0 = \operatorname{cis}(0 + k2\pi)$ for all $k \in \mathbb{Z}$

So $z^5 = \operatorname{cis}(k2\pi)$

$$\therefore\ z = [\operatorname{cis}(k2\pi)]^{\frac{1}{5}}$$
$$\therefore\ z = \operatorname{cis}\left(\frac{k2\pi}{5}\right) \quad \{\text{De Moivre's theorem}\}$$
$$\therefore\ z = \operatorname{cis} 0,\ \operatorname{cis}\left(\tfrac{2\pi}{5}\right),\ \operatorname{cis}\left(\tfrac{4\pi}{5}\right),\ \operatorname{cis}\left(\tfrac{6\pi}{5}\right),$$
$$\operatorname{cis}\left(\tfrac{8\pi}{5}\right) \quad \{\text{letting } k = 0, 1, 2, 3, 4\}$$



b
$$w = \operatorname{cis}\left(\tfrac{2\pi}{5}\right)$$
$$\text{Now } w^5 = 1$$
$$\therefore\ w^5 - 1 = 0$$
$$\therefore\ (w-1)(w^4 + w^3 + w^2 + w + 1) = 0$$
$$\therefore\ w^4 + w^3 + w^2 + w + 1 = 0 \quad \{w \neq 1\}$$

37 $z^2 - z + 1 + i = 0$

$$\therefore\ z = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1+i)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{1 - 4 - 4i}}{2}$$
$$= \frac{1 \pm \sqrt{-3-4i}}{2}$$

Let $a + bi = \sqrt{-3-4i}$, $a, b \in \mathbb{R}$

$\therefore \quad (a+bi)^2 = -3-4i$

$\therefore \quad a^2 + 2abi - b^2 = -3-4i$

Equating real and imaginary parts,

$a^2 - b^2 = -3 \quad (1) \qquad\qquad 2ab = -4 \quad (2)$

From (2), $ab = -2$, and so $b = -\dfrac{2}{a}$

Substituting into (1): $a^2 - \left(\dfrac{-2}{a}\right)^2 = -3$

$$a^2 - \frac{4}{a^2} + 3 = 0$$
$$\therefore \quad a^4 - 4 + 3a^2 = 0$$
$$(a^2+4)(a^2-1) = 0$$
$$\therefore \quad a = \pm 1 \quad \{a \in \mathbb{R}\}$$
$$\therefore \quad b = \mp 2$$
$$\therefore \quad \sqrt{3-4i} = 1-2i \quad \{a > 0\}$$

So, $z = \dfrac{1 \pm (1-2i)}{2} = i$ or $1 - i$

38 Let $z = a + bi$ and $w = c + di$.

Then $(z+w)^* = (a+bi+c+di)^*$
$= ((a+c)+(b+d)i)^*$
$= (a+c)-(b+d)i$
$= (a-bi)+(c-di)$
$= z^* + w^*$

39 Let $z_1 = |z_1| \operatorname{cis}\theta$ and $z_2 = |z_2| \operatorname{cis}\phi$.

Then $\dfrac{z_1}{z_2} = \dfrac{|z_1| \operatorname{cis}\theta}{|z_2| \operatorname{cis}\phi}$
$= \dfrac{|z_1|}{|z_2|} \operatorname{cis}(\theta - \phi) \quad$ {property of cis}

So $\arg\left(\dfrac{z_1}{z_2}\right) = \theta - \phi$
$= \arg(z_1) - \arg(z_2)$.

40 $\dfrac{z^2+3}{z^2-1} = k$

$\therefore \quad z^2 + 3 = k(z^2-1)$

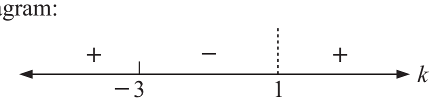
$\therefore \quad z^2 + 3 = kz^2 - k$

$\therefore \quad z^2(1-k) = -k-3$

$\therefore \quad z^2 = \dfrac{-k+3}{1-k}$

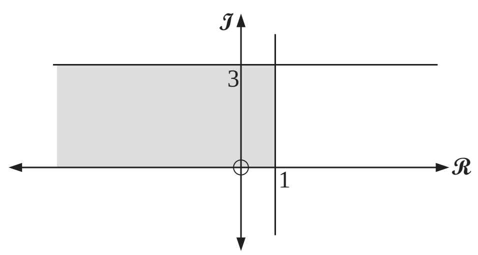
$\therefore$ the equation has imaginary roots if $\dfrac{-k-3}{1-k} < 0$

Sign diagram:

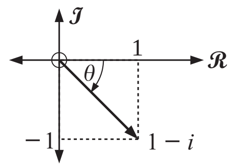


So, the equation has imaginary roots for $-3 < k < 1$.

41



42 $|1-i| = \sqrt{1^2 + (-1)^2}$
$= \sqrt{2}$



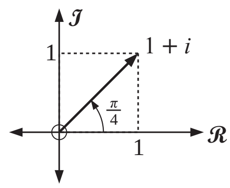
$\theta = \frac{\pi}{4}$ and so $\arg(1-i) = -\frac{\pi}{4}$

$\therefore \quad 1 - i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \quad (1-i)^{11} = \left[\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right]^{11}$
$= (\sqrt{2})^{11}\operatorname{cis}\left(-\frac{11\pi}{4}\right) \quad$ {De Moivre's theorem}
$= 32\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$
$= 32\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$
$= 32\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right]$
$= -32 - 32i$

43 $z = \dfrac{-1+5i}{2+3i}$
$= \left(\dfrac{-1+5i}{2+3i}\right)\left(\dfrac{2-3i}{2-3i}\right)$
$= \dfrac{-2+3i+10i-15i^2}{4-9i^2}$
$= \dfrac{13+13i}{13}$
$= 1 + i$

$|1+i| = \sqrt{1^2+1^2}$
$= \sqrt{2}$



$\arg(1+i) = \frac{\pi}{4}$

$\therefore \quad z = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

$\therefore \quad z^{12} = \left[\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right]^{12}$
$= (\sqrt{2})^{12}\operatorname{cis}\left(\frac{12\pi}{4}\right) \quad$ {De Moivre's theorem}
$= 64\operatorname{cis}(\pi)$
$= 64(\cos\pi + i\sin\pi)$
$= -64$

44 Let $z = r_1 \operatorname{cis}\theta$ and $w = r_2 \operatorname{cis}\phi$

Then $\dfrac{z}{w} = \dfrac{r_1 \operatorname{cis}\theta}{r_2 \operatorname{cis}\theta}$
$= \dfrac{r_1}{r_2}\operatorname{cis}(\theta - \phi) \quad$ {property of cis}

So $\left|\dfrac{z}{w}\right| = \dfrac{r_1}{r_2}$
$= \dfrac{|z|}{|w|}, \quad w \neq 0$

45 $|z-3| = |z-1|$

$\therefore \quad |(x+iy)-3| = |(x+iy)-1|$

$\therefore \quad |(x-3)+iy| = |(x-1)+iy|$

$\therefore \quad \sqrt{(x-3)^2+y^2} = \sqrt{(x-1)^2+y^2}$

$\therefore \quad (x-3)^2 + y^2 = (x-1)^2 + y^2$

$\therefore \quad x^2 - 6x + 9 = x^2 - 2x + 1$

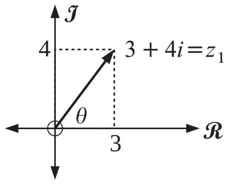
$\therefore \quad 4x = 8$

$\therefore \quad x = 2$

46 **a** $|z_1| = \sqrt{3^2+4^2}$

$= \sqrt{9+16}$

$= 5$



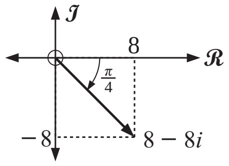
$\therefore \ \arg(z_1) = \arctan\left(\frac{4}{3}\right)$

≈ 0.927

$|z_2| = \sqrt{8^2+(-8)^2}$

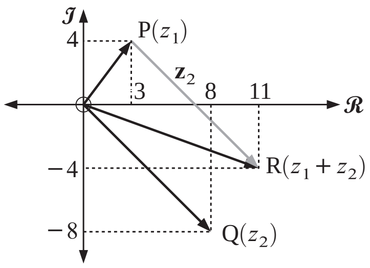
$= \sqrt{64+64}$

$= 8\sqrt{2}$



$\arg(z_2) = -\frac{\pi}{4}$

b



$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$

47 **a** $z^2 = z^*$

$\therefore \ (r \operatorname{cis}\theta)^2 = (r \operatorname{cis}\theta)^*$

$\therefore \ r^2 \operatorname{cis} 2\theta = r \operatorname{cis}(-\theta)$ {De Moivre's theorem}

$\therefore \ r^2 = r$ and $\operatorname{cis} 2\theta = \operatorname{cis}(-\theta)$

$\therefore \ \dfrac{\operatorname{cis} 2\theta}{\operatorname{cis}(-\theta)} = 1$

$\therefore \ \operatorname{cis} 3\theta = 1$

b $r^2 = r$

$\therefore \ r^2 - r = 0$

$\therefore \ r(r-1) = 0$

$\therefore \ r = 1 \ \{r > 0\}$

Now $\operatorname{cis} 3\theta = 1$

$\therefore \ 3\theta = 0 + 2k\pi, \ k \in \mathbb{Z}$

$\therefore \ \theta = \dfrac{2k\pi}{3}, \ k \in \mathbb{Z}$

$\therefore \ \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \quad \{0 \leqslant \theta \leqslant 2\pi\}$

$\operatorname{cis} 0 = 1$

$\operatorname{cis}\frac{2\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\operatorname{cis}\frac{4\pi}{3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

So, the non-zero solutions to $z^2 = z^*$ are $z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

48 $z = iz^*$

$\therefore \ x + iy = i(x - iy)$

$= ix + y$

$\therefore \ x = y$ {equating real and imaginary parts}

49 **a** $\operatorname{cis}\theta \operatorname{cis}\phi = e^{i\theta}e^{i\phi}$

$= e^{i\theta + i\phi}$

$= e^{i(\theta+\phi)}$

$= \operatorname{cis}(\theta+\phi)$

b $(r\operatorname{cis}\theta)^n = (re^{i\theta})^n$

$= r^n(e^{i\theta})^n$

$= r^n e^{in\theta}$

$= r^n \operatorname{cis} n\theta$

c $w = e^{i\left(\frac{2\pi}{5}\right)}$

$\therefore \ w^5 = \left(e^{i\left(\frac{2\pi}{5}\right)}\right)^5$

$= e^{2\pi i}$

$= \operatorname{cis} 2\pi$

$= 1$

So $w^5 - 1 = 0$

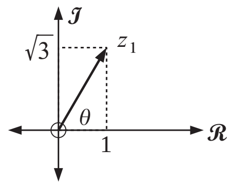
$\therefore \ (w-1)(1 + w + w^2 + w^3 + w^4) = 0$

$\therefore \ 1 + w + w^2 + w^3 + w^4 = 0 \ \{w \neq 1\}$

$\therefore \ 1 + w + w^2 + w^3 = -w^4$

$\therefore \ (1+w)(1+w^2) = -w^4$

50 **a** Let $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 + i$.



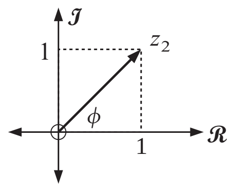
$|z_1| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \ \theta = \frac{\pi}{3}$

$\therefore \ z_1 = 2\operatorname{cis}\frac{\pi}{3}$



$|z_2| = \sqrt{1^2+1^2}$

$= \sqrt{2}$

$\cos\phi = \frac{1}{\sqrt{2}}, \quad \sin\phi = \frac{1}{\sqrt{2}}$

$\therefore \ \phi = \frac{\pi}{4}$

$\therefore \ z_2 = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$

So $z = \dfrac{1 + i\sqrt{3}}{1+i}$

$= \dfrac{2\operatorname{cis}\frac{\pi}{3}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}$

$= \left(\frac{2}{\sqrt{2}}\right)\operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sqrt{2}\operatorname{cis}\frac{\pi}{12}$

b **i** If $z^n \in \mathbb{R}$, $\arg(z^n) = k\pi, \ k \in \mathbb{Z}$

$z^n = \left(\sqrt{2}\operatorname{cis}\frac{\pi}{12}\right)^n$

$= (\sqrt{2})^n \operatorname{cis}\frac{n\pi}{12}$

$\therefore \ \arg(z^n) = \frac{n\pi}{12}$

So the smallest positive value of n which satisfies $\frac{n\pi}{12} = k\pi$ is $n = 12$.

ii If z^n is purely imaginary, $\arg(z^n) = \frac{\pi}{2} + k\pi$

$\therefore \ \frac{n\pi}{12} = \frac{\pi}{2} + k\pi$

The smallest positive value of n which satisfies $\frac{n\pi}{12} = \frac{\pi}{2} + k\pi$ is $n = 6$.

51 **a** $w = e^{i\left(\frac{2\pi}{3}\right)}$

$\therefore \ w^3 = \left(e^{i\left(\frac{2\pi}{3}\right)}\right)^3$

$= e^{2\pi i}$

$= \operatorname{cis} 2\pi$

$= 1$

$\therefore \ w^3 - 1 = 0$

$\therefore \ (w-1)(1 + w + w^2) = 0$

$\therefore \ 1 + w + w^2 = 0 \ \{w \neq 1\}$

b **i** $w^7 = w^3 w^3 w$

$= w$

ii $w^3 = 1$

$\therefore \ w^2 = \dfrac{1}{w}$

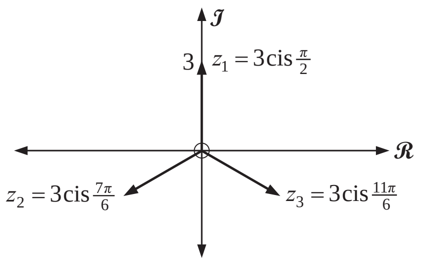
$\therefore \ w^{-1} = w^2$

iii $(1-w)^2 = 1-2w+w^2$
$= 1+w+w^2-3w$
$= -3w$

iv $\dfrac{1}{(1+w)^2} = \dfrac{1}{1+2w+w^2}$
$= \dfrac{1}{1+w+w^2+w}$
$= \dfrac{1}{w}$
$= w^{-1}$
$= w^2$ {part **ii**}

v $\dfrac{1+w^2}{1+w} = \dfrac{1+w+w^2-w}{1+w+w^2-w^2}$
$= \dfrac{-w}{-w^2}$
$= \dfrac{1}{w}$
$= w^{-1}$
$= w^2$

52 **a** Let $z^3 = -27i$
$\therefore\ z^3 = 27\,\text{cis}\left(\frac{3\pi}{2}\right)$
$\therefore\ z^3 = 27\,\text{cis}\left(\frac{3\pi}{2}+k2\pi\right),\ k\in\mathbb{Z}$
$\therefore\ z = 3\,\text{cis}\left(\frac{\pi}{2}+\frac{k2\pi}{3}\right)$ {De Moivre's theorem}
$\therefore\ z = 3\,\text{cis}\,\frac{\pi}{2},\ 3\,\text{cis}\,\frac{7\pi}{6},\ 3\,\text{cis}\,\frac{11\pi}{6}$ $\{k=0,\,1,\,2\}$



b Let $z_1 = 3\,\text{cis}\,\frac{\pi}{2},\ z_2 = 3\,\text{cis}\,\frac{7\pi}{6}$
$z_3 = 3\,\text{cis}\,\frac{11\pi}{6}$
$z_2z_3 = 3\,\text{cis}\,\frac{7\pi}{6}\times 3\,\text{cis}\,\frac{11\pi}{6}$
$= 9\,\text{cis}\left(\frac{18\pi}{6}\right)$
$= -9$
${z_1}^2 = \left(3\,\text{cis}\,\frac{\pi}{2}\right)^2$
$= 9\,\text{cis}\,\pi$
$= -9$ and so $z_2z_3 = {z_1}^2$

c $z_1z_2z_3 = z_1({z_1}^2)$ {from **b**}
$= {z_1}^3$
$= -27i$

53 $\cos 3\theta + i\sin 3\theta$
$= \text{cis}\,3\theta$
$= (\text{cis}\,\theta)^3$ {De Moivre's theorem}
$= (\cos\theta + i\sin\theta)^3$
$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$
$= [\cos^3\theta - 3\cos\theta\sin^2\theta] + i[3\cos^2\theta\sin\theta - \sin^3\theta]$

Equating imaginary parts,
$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$

54 **a** $z^n + \dfrac{1}{z^n}$
$= z^n + z^{-n}$
$= (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}$
$= \text{cis}(n\theta) + \text{cis}(-n\theta)$ {De Moivre's theorem}
$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$
$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$
$= 2\cos(n\theta)$

b $\left(z+\dfrac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \dfrac{4}{z^2} + \dfrac{1}{z^4}$
$\therefore\ \left(z+\dfrac{1}{z}\right)^4 = \left(z^4+\dfrac{1}{z^4}\right) + 4\left(z^2+\dfrac{1}{z^2}\right) + 6$
$\therefore\ (2\cos\theta)^4 = (2\cos 4\theta) + 4\,(2\cos 2\theta) + 6$
$\therefore\ 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$
$\therefore\ \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

55 $T_{r+1} = \binom{12}{r}(2x)^{12-r}\left(\dfrac{-1}{x^2}\right)^r$
$= \binom{12}{r}2^{12-r}x^{12-r}(-1)^r x^{-2r}$
$= \binom{12}{r}2^{12-r}(-1)^r x^{12-3r}$

a $12 - 3r = 3$
$\therefore\ r = 3$
$\therefore\ T_4 = \binom{12}{3}2^{12-3}(-1)^3x^3$
$= -112\,640x^3$

The coefficient of x^3 is $-112\,640$.

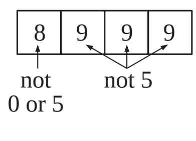
b $12 - 3r = 0$
$\therefore\ r = 4$
$\therefore\ T_5 = \binom{12}{4}2^{12-4}(-1)^4x^0$
$= 126\,720$

The constant term is 126 720.

56 $\binom{n}{r} + \binom{n}{r-1} = \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-(r-1))!}$
$= \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n+1-r)!}$
$= \dfrac{n!(n+1-r)}{r!(n+1-r)!} + \dfrac{n!r}{r!(n+1-r)!}$
$= \dfrac{n!(n+1-r)+n!r}{r!(n+1-r)!}$
$= \dfrac{n!(n+1)-n!r+n!r}{r!(n+1-r)!}$
$= \dfrac{(n+1)!}{r!\,((n+1)-r)!}$
$= \binom{n+1}{r}$ for all $n,\ r\in\mathbb{Z}^+,\ r\leqslant n$

57 There are 12 numbers up for selection and we choose 3.
$\therefore$ the total number of combinations is $C_3^{12} = 220$.

58 There are 9000 integers between 1000 and 9999 inclusive.
Integers that do *not* contain a 5:



$\therefore$ there are $8 \times 9 \times 9 \times 9 = 5832$ integers that do not contain a 5.

$\therefore$ number of integers that *do* contain a 5 $= 9000 - 5832 = 3168$.

59 $T_{r+1} = \binom{n}{r}(3x)^{n-r}(2)^r$

$= \binom{n}{r} \times 3^{n-r} \times 2^r \times x^{n-r}$

Coefficient of x^3: $n - r = 3$

$\therefore \quad r = n-3$

$T_{n-2} = \binom{n}{n-3} \times 3^3 \times 2^{n-3} \times x^3$

Coefficient of x: $n - r = 1$

$\therefore \quad r = n - 1$

$T_n = \binom{n}{n-1} \times 3 \times 2^{n-1} \times x$

Now $\left[\binom{n}{n-1} \times 3 \times 2^{n-1}\right] \times 21 = \binom{n}{n-3} \times 3^3 \times 2^{n-3}$

$$\therefore \quad \frac{n!}{(n-1)!\,(n-(n-1))!} \times 3^2 \times 2^{n-1} \times 7 = \frac{n!}{(n-3)!\,(n-(n-3))!} \times 3^3 \times 2^{n-3}$$

$$\therefore \quad n \times 3^2 \times 2^{n-1} \times 7 = \frac{n(n-1)(n-2)}{6} \times 3^3 \times 2^{n-3}$$

$$\therefore \quad n \times 2^2 \times 7 = \frac{n(n-1)(n-2)}{6} \times 3$$

$\therefore \quad 56n = n(n-1)(n-2)$

$\therefore \quad 56n = n(n^2 - 3n + 2)$

$\therefore \quad 56n = n^3 - 3n^2 + 2n$

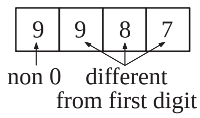
$\therefore \quad n^3 - 3n^2 - 54n = 0$

$\therefore \quad n(n^2 - 3n - 54) = 0$

$\therefore \quad n(n-9)(n+6) = 0$

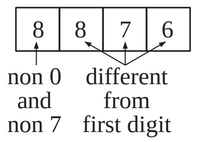
$\therefore \quad n = 9 \quad \{n \in \mathbb{Z}^+\}$

60 **a**



$\therefore$ there are $9 \times 9 \times 8 \times 7 = 4536$ numbers

b Numbers that do *not* have a "7" as one of the four digits:



$\therefore$ there are $8 \times 8 \times 7 \times 6 = 2688$ numbers that do not contain a "7".

$\therefore$ there are $4536 - 2688 = 1848$ numbers that *do* contain a "7".

61 $T_{r+1} = \binom{9}{r}(kx)^{9-r}\left(\frac{1}{\sqrt{x}}\right)^r$

$= \binom{9}{r} k^{9-r} x^{9-r} \frac{1}{x^{\frac{r}{2}}}$

$= \binom{9}{r} k^{9-r} x^{9-\frac{3r}{2}}$

For the constant term, $9 - \frac{3r}{2} = 0$

$\therefore \quad \frac{3r}{2} = 9$

$\therefore \quad r = 6$

$T_7 = \binom{9}{6} k^3 x^0$

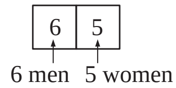
$\therefore \quad 84k^3 = -10\frac{1}{2}$

$\therefore \quad k^3 = -\frac{1}{8}$

$k = -\frac{1}{2}$

62 **a** Choose 2 people from 11. There are $\binom{11}{2} = 55$ ways of doing this.

b



So, there are $6 \times 5 = 30$ handshakes between a man and a woman.

63 6 of the 24 upstairs seats are taken. We hence choose 18 people from the $48-(8+6) = 34$ remaining passengers to sit upstairs; the rest sit downstairs.

$\therefore$ there are $\binom{34}{18} = 2\,203\,961\,430$ ways.

64 $(x+2)(1-x)^{10}$

$$= (x+2)\left(1^{10} + \binom{10}{1}1^9(-x) + ... + \binom{10}{4}1^6(-x)^4 + \binom{10}{5}1^5(-x)^5 + ...\right)$$

$$= (x+2)\left(1 - 10x + ... + \binom{10}{4}x^4 - \binom{10}{5}x^5 + ...\right)$$

So, the terms containing x^5 are $\binom{10}{4}x^5$ and $-2\binom{10}{5}x^5$.

$\therefore$ the coefficient of x^5 is $\binom{10}{4} - 2\binom{10}{5} = -294$

65 If all pairs of points defined different lines, there would be $\binom{11}{2} = 55$ lines.

There are $\binom{4}{2} = 6$ ways of choosing a pair of points from the 4 collinear points, of which we include one.

So the total number of lines is $55 - 6 + 1 = 50$.

66 $(1+x)^n = \binom{n}{0} \times 1^n \times x^0 + \binom{n}{1} \times 1^{n-1} \times x^1 + \binom{n}{2} \times 1^{n-2} \times x^2 + ... + \binom{n}{n} \times 1^0 \times x^n$

Now set $x = 1$:

$(1+1)^n = \binom{n}{0} \times 1^n \times 1^0 + \binom{n}{1} \times 1^{n-1} \times 1^1 + \binom{n}{2} \times 1^{n-2} \times 1^2 + ... + \binom{n}{n} \times 1^0 \times 1^n$

$\therefore \quad 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n}$

$\therefore \quad 2^n = \sum_{r=0}^{n} \binom{n}{r}$

67 If the mathematics exams are consecutive, then we can treat them as one subject, giving 7 exams to order.

This pair of mathematics subjects has 2! orderings, so there are $7! \times 2!$ schedules that cannot be used.

There are 8! total orderings.

So the teacher can choose from $8! - (7! \times 2!) = 30\,240$ schedules.

68 Consider the x^n term in: $(1+x)^{2n} = (1+x)^n(1+x)^n$

On the LHS, $T_{n+1} = \binom{2n}{n}1^n x^n$, so x^n has coefficient $\binom{2n}{n}$.

On the RHS, we have

$(1+x)^n(1+x)^n$

$$= \left[\binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]$$

$$\times \left[\binom{n}{0} + \binom{n}{1}x + ... + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n\right]$$

$\therefore$ the coefficient of x^n

$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + ... + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}$

$= \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + ... + \binom{n}{n-1}\binom{n}{n-1} + \binom{n}{n}\binom{n}{n}$

$\{\binom{n}{r} = \binom{n}{n-r}\}$

$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2$

Equating coefficients of x^n,

$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n}^2$

69 $n\binom{n-1}{r-1} = n\dfrac{(n-1)!}{(r-1)!\,(n-1-(r-1))!}$

$= \dfrac{n!}{(r-1)!(n-r)!}$

$= r\dfrac{n!}{r!(n-r)!}$

$= r\binom{n}{r}$ for $n, r \in \mathbb{Z}^+$, $n \geqslant r$.

70 Each person is either in or out of the committee, so there are $2^{12} = 4096$ possible committees.

There are $\binom{12}{1} = 12$ 1-member committees, and $\binom{12}{0} = 1$ 0-member committees.

Hence there are $4096 - (12+1) = 4083$ committees with at least two members.

71 $T_{r+1} = \binom{9}{r}(\sqrt{x})^{9-r}\left(\dfrac{b}{x}\right)^r$

$= \binom{9}{r} \times x^{\frac{9-r}{2}} \times b^r \times \dfrac{1}{x^r}$

$= \binom{9}{r} \times b^r \times x^{\frac{9-3r}{2}}$

For the coefficient of x^{-3}, $\dfrac{9-3r}{2} = -3$

$\therefore\ r = 5$

$T_6 = \binom{9}{5} b^5 x^{-3}$

Now, x^{-3} has coefficient -4032, so $\binom{9}{5} b^5 = -4032$

$\therefore\ b^5 = -32$

$\therefore\ b = -2$

72 **a** $\binom{16}{3} = 560$ triangles

b There are 5 possible points on the circle and $\binom{11}{2}$ possible points within the circle.

$\therefore$ there are $5 \times \binom{11}{2} = 275$ triangles.

c There are $\binom{5}{3} = 10$ triangles with all vertices on the circle. Hence there are $560 - 10 = 550$ triangles with at least one of the vertices within the circle.

73 P_n is: "$1\times 2 + 2\times 5 + 3 \times 8 + ... + n(3n-1) = n^2(n+1)$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1 \times 2 = 2$ and RHS $= 1^2(1+1) = 2$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$1 \times 2 + 2 \times 5 + 3 \times 8 + ... + k(3k-1) = k^2(k+1)$

Thus $1 \times 2 + 2 \times 5 + 3 \times 8 + ... + k(3k-1) + (k+1)(3(k+1)-1)$

$= k^2(k+1) + (k+1)(3(k+1)-1)$ {using P_k}

$= (k+1)\left(k^2 + 3(k+1) - 1\right)$

$= (k+1)(k^2 + 3k + 2)$

$= (k+1)(k+1)(k+2)$

$= (k+1)^2([k+1]+1)$

Thus P_{k+1} is true when P_k is true. Since P_1 is true, P_n is true for $n \in \mathbb{Z}^+$.

{Principle of mathematical induction}

74 P_n is: "$5n^3 - 3n^2 - 2n$ is divisible by 6" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $5 \times 1^3 - 3 \times 1^2 - 2 \times 1 = 0$
$= 0 \times 6$

$\therefore$ P_1 is true.

(2) If P_k is true, then $5k^3 - 3k^2 - 2k = 6A$ where A is an integer.

$5(k+1)^3 - 3(k+1)^2 - 2(k+1)$

$= 5(k^3 + 3k^2 + 3k + 1) - 3(k^2 + 2k + 1) - 2(k+1)$

$= (5k^3 - 3k^2 - 2k) + 15k^2 + 9k$

$= 6A + 3k(5k+3)$ {using P_k}

$= 6A + 3(2B)$, $B \in \mathbb{Z}$

{$k(5k+3)$ is divisible by 2, since either k is divisible by 2, or k is odd $\Rightarrow 5k+3$ is divisible by 2}

$= 6(A+B)$, where $A, B \in \mathbb{Z}$

So, P_{k+1} is true.

Thus P_{k+1} is true whenever P_k is true, and P_1 is true.

$\therefore$ P_n is true for $n \in \mathbb{Z}^+$.

{Principle of mathematical induction}

75 **a** Suppose $n = 1$. $9^1 + b$ is divisible by 8 if $b = 7$
$\{0 < b \leqslant 9\}$.

So, $b = 7$.

b P_n is: "$9^n + 7$ is divisible by 8" for $n \in \mathbb{Z}^+$

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $9^1 + 7 = 16 = 8 \times 2$

$\therefore$ P_1 is true.

(2) If P_k is true, then $9^k + 7 = 8A$ where $A \in \mathbb{Z}^+$.

$9^{k+1} + 7 = 9 \times 9^k + 7$

$= 9 \times (9^k + 7) - 9 \times 7 + 7$

$= 9 \times 8A - 9 \times 7 + 7$ {using P_k}

$= 9 \times 8A - 8 \times 7$

$= 8(9A - 7)$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$.

{Principle of mathematical induction}

76 P_n is: "$1 + 2 \times 2^1 + 3 \times 2^2 + ... + n \times 2^{n-1} = (n-1)2^n + 1$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 1$ and RHS $= (1-1)2^1 + 1 = 1$

$\therefore$ P_1 is true.

(2) If P_k is true, then

$1 + 2 \times 2^1 + 3 \times 2^2 + ... + k \times 2^{k-1} = (k-1)2^k + 1$

So $1 + 2 \times 2^1 + 3 \times 2^2 + ... + k \times 2^{k-1} + (k+1)2^k$

$= (k-1)2^k + 1 + (k+1) \times 2^k$ {using P_k}

$= k \times 2^k - 2^k + 1 + k \times 2^k + 2^k$

$= 2 \times k \times 2^k + 1$

$= ([k+1] - 1)\,2^{k+1} + 1$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

77 **a** $4x^2 = x^2 + 2x^2 + x^2$

$\geqslant x^2 + 2x + 1 \quad \{\text{if } x \geqslant 1,\ x^2 \geqslant x, \text{ and } x^2 \geqslant 1\}$

$\geqslant (x+1)^2$

b P_n is "$4^n \geqslant 3n^2$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 4^1 = 4$ and RHS $= 3 \times 1^2 = 3$
$4 \geqslant 3$, so P_1 is true,

(2) If P_k is true, $4^k \geqslant 3k^2$

Now, $4^{k+1} = 4 \times 4^k$

$\geqslant 4(3k^2) \quad \{\text{using } P_k\}$

$\geqslant 3(4k^2)$

$\geqslant 3(k+1)^2 \quad \{\text{using } \mathbf{a},\ k \geqslant 1\}$

So P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

78 P_n is "$n^3 + 2n$ is divisible by 3" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, $1^3 + 2 \times 1 = 3 = 3 \times 1$
$\therefore$ P_1 is true.

(2) If P_k is true, $k^3 + 2k = 3A$ where $A \in \mathbb{Z}$

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + 3k^2 + 3k + 3$

$= 3A + 3(k^2 + k + 1) \quad \{\text{using } P_k\}$

$= 3(A + k^2 + k + 1)$

$\therefore$ P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

79 P_n is: "$3^n > n^2 + n$" for $n \in \mathbb{Z}^+$.

Proof: (By the principle of mathematical induction)

(1) If $n = 1$, LHS $= 3^1 = 3$ and RHS $= 1^2 + 1 = 2$
$\therefore$ P_1 is true.

(2) If P_k is true, $3^k > k^2 + k$

$3^{k+1} = 3 \times 3^k$

$> 3(k^2 + k) \quad \{\text{using } P_k\}$

$> 3k^2 + 3k$

$> 2k^2 + k^2 + 3k$

$> 2 + k^2 + 3k \quad \{k \geqslant 1\}$

$> k^2 + 2k + 1 + k + 1$

$> (k+1)^2 + (k+1)$

So, P_{k+1} is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of mathematical induction}

SOLUTIONS TO TOPIC 2 (FUNCTIONS AND EQUATIONS)

1 A *function* is a relation in which no two different ordered pairs have the same x-coordinate. A graph of a relation is a function if all possible vertical lines on the graph cut the relation no more than once.

2 Domain $= \{x \mid x \geqslant -1,\ x \neq 1\}$

3 Domain $= \{x \mid x > 2\}$ Range $= \{y \mid y \in \mathbb{R}\}$

4 **a** $f(g(x)) = f(4 - x)$

$= 3(4 - x) + 1$

$= -3x + 13$

b $(g \circ f)(x) = g(3x + 1)$

$= 4 - (3x + 1)$

$= 3 - 3x$

So $(g \circ f)(-4) = 3 - 3(-4)$

$= 15$

c f is $y = 3x + 1$, so f^{-1} is $x = 3y + 1$

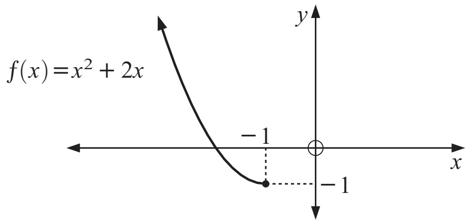
$\therefore$ $y = \dfrac{x-1}{3}$

$\therefore$ $f^{-1}(x) = \dfrac{x-1}{3}$

so $f^{-1}(\frac{1}{2}) = \dfrac{\frac{1}{2} - 1}{3}$

$= -\frac{1}{6}$

5



f is $y = x^2 + 2x$, $x \in \,]-\infty, -1]$

so f^{-1} is $x = y^2 + 2y$, $y \in \,]-\infty, -1]$

$\therefore$ $x + 1 = y^2 + 2y + 1$

$\therefore$ $x + 1 = (y+1)^2$

$\therefore$ $\pm\sqrt{x+1} = y + 1$

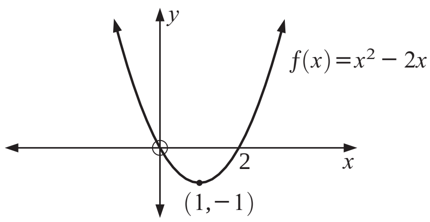
$y = -\sqrt{x+1} - 1 \quad \{\text{as } y \leqslant -1\}$

so $f^{-1}(x) = -\sqrt{x+1} - 1$

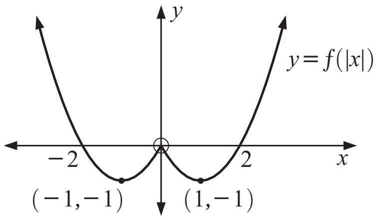
Any point of $f(x)$ which lies on the line $y = x$ will be an invariant point.

$\therefore$ $(-1, -1)$ is an invariant point.

6 **a**



b **i**



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