40 **a**
$$\begin{aligned} f(x) &= 2\cos x - \sin x + 2x\sin x \\ \therefore\ f'(x) &= -2\sin x - \cos x + 2\sin x + 2x\cos x \\ &= 2x\cos x - \cos x \\ &= (2x-1)\cos x \end{aligned}$$

b $f'(x) = 0$ when $2x - 1 = 0$ or $\cos x = 0$

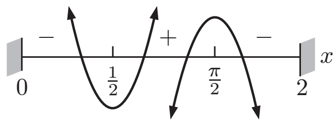
$\therefore$ for $0 \leqslant x \leqslant 2$, $x = \frac{1}{2}$ or $x = \frac{\pi}{2}$

$f'(\frac{1}{4}) = -\frac{1}{2}\cos\frac{1}{4} < 0$

$f'(1) = \cos 1 > 0$

$f'(1.9) = 2.8 \times \cos 1.9 \approx -0.9052 < 0$

Sign diagram for $f'(x)$:

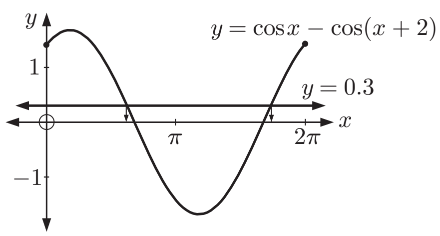


Now $f(\frac{1}{2}) = 2\cos\frac{1}{2} - \sin\frac{1}{2} + \sin\frac{1}{2}$
≈ 1.755

and $f(2) = 2\cos 2 - \sin 2 + 4\sin 2$
$= 2\cos 2 + 3\sin 2$
≈ 1.896

$\therefore$ the smallest value is ≈ 1.76 when $x = \frac{1}{2}$.

41
$$\begin{aligned} \int_a^{a+2} \sin x\,dx &= 0.3 \\ \therefore\ [-\cos x]_a^{a+2} &= 0.3 \\ \therefore\ -\cos(a+2) + \cos a &= 0.3 \\ \therefore\ \cos a - \cos(a+2) &= 0.3 \end{aligned}$$



So, $x = a \approx 1.96$ or 5.46 {using technology}

SOLUTIONS TO EXAMINATION PRACTICE SET 1

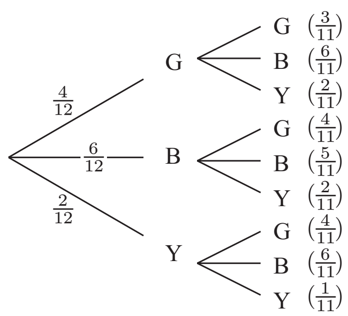
1 **a** **i**
$$\begin{aligned} y &= 3e^{x^2} + 3(2-x) \\ &= 3e^{x^2} + 6 - 3x \\ \therefore\ \frac{dy}{dx} &= 3e^{x^2}(2x) + 0 - 3 \\ &= 6xe^{x^2} - 3 \end{aligned}$$

ii
$$\begin{aligned} y &= \ln(2x+5) \\ \therefore\ \frac{dy}{dx} &= \frac{1}{2x+5} \times 2 \\ \therefore\ \frac{dy}{dx} &= \frac{2}{2x+5} \end{aligned}$$

b **i**
$$\begin{aligned} &\int \sin(\pi - x)\,dx \\ &= \left(\tfrac{1}{-1}\right)(-\cos(\pi - x)) + c \\ &= \cos(\pi - x) + c \end{aligned}$$

ii
$$\begin{aligned} &\int x(1+x)\,dx \\ &= \int (x + x^2)\,dx \\ &= \frac{x^2}{2} + \frac{x^3}{3} + c \end{aligned}$$

2



$$\begin{aligned} &\text{P(different colours)} \\ &= 1 - \text{P(same colours)} \\ &= 1 - \left(\tfrac{4}{12} \times \tfrac{3}{11} + \tfrac{6}{12} \times \tfrac{5}{11} + \tfrac{2}{12} \times \tfrac{1}{11}\right) \\ &= 1 - \tfrac{44}{12 \times 11} \\ &= 1 - \tfrac{1}{3} \\ &= \tfrac{2}{3} \end{aligned}$$

3 **a** $\log_7 A^3 = 3\log_7 A$
$= 3x$

b $\log_7 \dfrac{1}{\sqrt{A}} = \log_7 A^{-\frac{1}{2}}$
$= -\frac{1}{2}\log_7 A$
$= -\frac{1}{2}x$

c Let $\log_{49} A = m$
$\therefore\ A = 49^m = 7^{2m}$
$\therefore\ \log_7 A = 2m$
So, $\log_{49} A = \frac{1}{2}\log_7 A = \frac{1}{2}x$

4 **a**
$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} + (t-1)\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 1 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \{\text{where } s = t - 1\} \end{aligned}$$

$\therefore$ **A** is the same line.

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 4 \\ -1 \end{pmatrix} + (t+2)\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -7 \\ 4 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \{\text{where } s = t + 2\} \end{aligned}$$

$\therefore$ **B** is the same line.

C is not the same line as the direction vector is $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ rather than $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{1}{3}t\begin{pmatrix} 12 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + s\begin{pmatrix} 12 \\ -3 \end{pmatrix} \quad \{\text{where } s = \tfrac{1}{3}t\} \end{aligned}$$

$\therefore$ **D** is the same line.

b $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$ has length $\sqrt{0 + 9 + 16} = \sqrt{25} = 5$

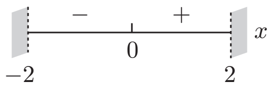
So, the velocity vector is $800 \times \frac{1}{5}\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = 160\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$.

The position vector is $\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 0.05 \end{pmatrix} + 160t\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$.

5 **a** Vertical asymptotes occur when $4 - x^2 = 0$
$\therefore\ x^2 = 4$
$\therefore$ the vertical asymptotes are $x = -2$ and $x = 2$.

b
$$\begin{aligned} f(x) &= \frac{4}{\sqrt{4 - x^2}} = 4(4 - x^2)^{-\frac{1}{2}} \\ \therefore\ f'(x) &= 4\left(-\tfrac{1}{2}\right)\left(4 - x^2\right)^{-\frac{3}{2}}(-2x) \\ &= 4x(4 - x^2)^{-\frac{3}{2}} \end{aligned}$$

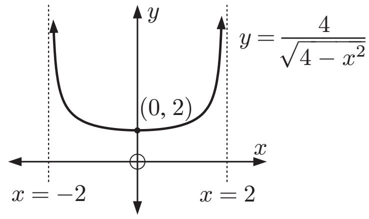
$f'(x)$ has sign diagram:



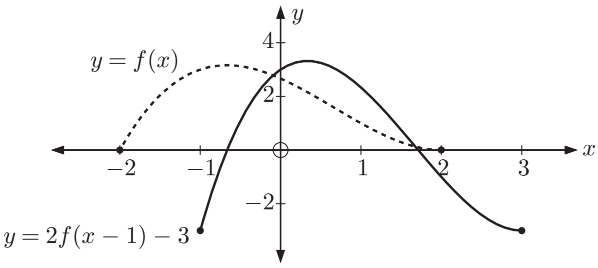
Now $f(0) = \frac{4}{\sqrt{4}} = 2$

$\therefore$ there is a local minimum at (0, 2).

c



6 To obtain $y = 2f(x-1) - 3$ we stretch vertically with dilation factor 2, then translate through $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.



7
- $a > 0$ as the parabola opens upwards
- c is the y-intercept, so $c > 0$
- The axis of symmetry is $x = \dfrac{-b}{2a}$ and so $\dfrac{-b}{2a} < 0$

  $\therefore$ since $a > 0$, $-b < 0$

  $\therefore$ $b > 0$
- $b^2 - 4ac = 0$ as the graph touches the x-axis

expression	*positive*	*negative*	*zero*
a	yes	no	no
b	yes	no	no
c	yes	no	no
$b^2 - 4ac$	no	no	yes

8 **a** g is $y = e^{-x}$

$\therefore$ g^{-1} is $x = e^{-y}$

$\therefore$ $-y = \ln x$

$\therefore$ $y = -\ln x$

$\therefore$ $g^{-1}(x) = -\ln x$

b $(f \circ g)(x) = f(g(x))$

$= f(e^{-x})$

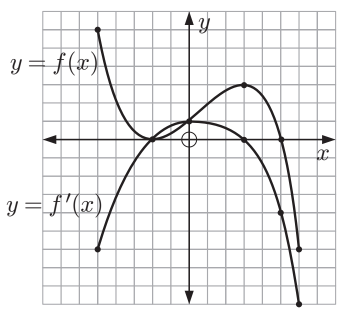
$= e^{-x} + \dfrac{1}{e^{-x}}$

$= e^{-x} + e^{x}$

9 At $x = -5$, $\dfrac{dy}{dx} \approx -6$ At $x = 3$, $\dfrac{dy}{dx} = 0$

At $x = -2$, $\dfrac{dy}{dx} = 0$ At $x = 5$, $\dfrac{dy}{dx} \approx -4$

At $x = 0$, $\dfrac{dy}{dx} \approx 1$ At $x = 6$, $\dfrac{dy}{dx} \approx -9$



10 **a** $kx^2 - 7x + 4 = 0$ has $\Delta = b^2 - 4ac$

$= 49 - 4(k)(4)$

$= 49 - 16k$

There is exactly one solution if $\Delta = 0$

$\therefore$ $49 - 16k = 0$

$\therefore$ $16k = 49$

$\therefore$ $k = \frac{49}{16}$

b $-1 \leqslant \sin(2x - \pi) \leqslant 1$

$\therefore$ $-3 \leqslant 3\sin(2x - \pi) \leqslant 3$

$\therefore$ $3\sin(2x - \pi) = n$ has no solutions if $n < -3$ or $n > 3$

c $y = 2\cos(4x + \pi)$

$\therefore$ $\dfrac{dy}{dx} = -2\sin(4x + \pi) \times 4$

$= -8\sin(4x + \pi)$

When $x = \frac{\pi}{8}$, $\dfrac{dy}{dx} = -8\sin(\frac{3\pi}{2}) = -8(-1) = 8$

So, the tangent has equation $y - 0 = 8(x - \frac{\pi}{8})$

which is $y = 8x - \pi$.

11 **a** $|\mathbf{a}| = \sqrt{(-2)^2 + 6^2} = \sqrt{4 + 36} = \sqrt{40}$

b $2\mathbf{a} - 3\mathbf{b} = \begin{pmatrix} -4 \\ 12 \end{pmatrix} - \begin{pmatrix} 12 \\ -15 \end{pmatrix} = \begin{pmatrix} -16 \\ 27 \end{pmatrix}$

c $\mathbf{a} \bullet \mathbf{b} = (-8) + (-30) = -38$

d $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $t \in \mathbb{R}$

12 **a** $2x^2 - 3mx + 8 = 0$ has exactly one solution for x if $\Delta = 0$

$\therefore$ $(-3m)^2 - 4(2)(8) = 0$

$\therefore$ $9m^2 = 64$

$\therefore$ $m^2 = \frac{64}{9}$

$\therefore$ $m = \pm\frac{8}{3}$

But $m > 0$, so $m = \frac{8}{3}$

b Consider $h : x \mapsto \sqrt{4x + 9}$ where $x \geqslant -2\frac{1}{4}$, $y \geqslant 0$

h^{-1} is given by $x = \sqrt{4y + 9}$ where $x \geqslant 0$, $y \geqslant -2\frac{1}{4}$

Rearranging, $x^2 = 4y + 9$

$\therefore$ $y = \dfrac{x^2 - 9}{4}$

So, $h^{-1}(x) = \dfrac{x^2 - 9}{4}$ for $x \geqslant 0$

and $h^{-1}(10) = \dfrac{10^2 - 9}{4} = \frac{91}{4}$

13

	Girls	*Boys*	*Total*
Computer Games	10	12	22
Sport	34	24	58
Total	44	36	80

a P(prefers sport) $= \frac{58}{80} = \frac{29}{40}$

b P(prefers sport | a girl) $= \frac{34}{44} = \frac{17}{22}$

14 $(ax - 3)^4$

$= (ax)^4 + 4(ax)^3(-3) + 6(ax)^2(-3)^2 + 4(ax)(-3)^3 + (-3)^4$

$= a^4x^4 - 12a^3x^3 + 54a^2x^2 - 108ax + 81$

15 **a** $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{AB} = \mathbf{C}$, where $\mathbf{A}$ is 3×2, $\mathbf{B}$ is $2 \times n$, $\mathbf{C}$ is $3 \times n$

i $\mathbf{B}$ is $2 \times n$ and $\mathbf{C}$ is $3 \times n$, for $n = 1, 2, 3, 4, ...$

ii If $\mathbf{B} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ then

$$\mathbf{C} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 8 \end{pmatrix}$$

b
$$\begin{aligned}(\mathbf{A}+2\mathbf{I})(\mathbf{A}-3\mathbf{I}) &= \mathbf{A}^2 + 2\mathbf{IA} - 3\mathbf{AI} - 6\mathbf{I}^2 \\ &= \mathbf{A}^2 + 2\mathbf{A} - 3\mathbf{A} - 6\mathbf{I} \\ &= \mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}\end{aligned}$$

If $(\mathbf{A}+2\mathbf{I})(\mathbf{A}-3\mathbf{I}) = \mathbf{O}$

then $\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I} = \mathbf{O}$

$\therefore \ \mathbf{A}(\mathbf{A}-\mathbf{I}) = 6\mathbf{I}$ {form $\mathbf{AB} = 6\mathbf{I}$}

$\therefore \ \mathbf{A}\frac{1}{6}(\mathbf{A}-\mathbf{I}) = \mathbf{I}$

$\therefore \ \mathbf{A}^{-1} = \frac{1}{6}(\mathbf{A}-\mathbf{I})$

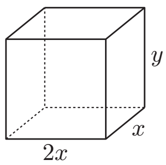
16 **a** **i** Volume = length × breadth × depth

$= 2x(x)y$

$= 2x^2y$

$\therefore \ 2x^2y = 9$ and so $y = \dfrac{9}{2x^2}$ .... $(*)$

ii Total surface area, $A = 2(2xx + xy + 2xy)$



$\therefore \ A = 2(2x^2 + 3xy)$

$\therefore \ A = 4x^2 + 6xy$

$\therefore \ A = 4x^2 + 6x\left(\dfrac{9}{2x^2}\right)$ {from $(*)$}

$\therefore \ A = \left(4x^2 + \dfrac{27}{x}\right)$ m^2

iii Now $A = 4x^2 + 27x^{-1}$

$\therefore \ \dfrac{dA}{dx} = 8x - 27x^{-2} = 8x - \dfrac{27}{x^2} = \dfrac{8x^3 - 27}{x^2}$

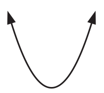
$\therefore \ \dfrac{dA}{dx} = 0$ when $8x^3 - 27 = 0$

$\therefore \ x^3 = \frac{27}{8}$

$\therefore \ x = \frac{3}{2}$

Now $\dfrac{d^2A}{dx^2} = 8 + 54x^{-3} = 8 + \dfrac{54}{x^3}$

$\therefore \ \dfrac{d^2A}{dx^2} > 0$ for all $x > 0$



is the shape of A against x.

$\therefore$ the minimum of A occurs when $x = 1\frac{1}{2}$

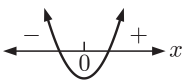
and $y = \dfrac{9}{2(\frac{3}{2})^2} = \dfrac{9}{\frac{9}{2}} = 2$

$\therefore$ the tank is $1\frac{1}{2}$ m by 3 m by 2 m.

b **i** $f(x) = 1 - \dfrac{1}{1+x^2} = 1 - (1+x^2)^{-1}$

$\therefore \ f'(x) = 0 + (1+x^2)^{-2}(2x) = \dfrac{2x}{(1+x^2)^2}$

$f'(x)$ has sign diagram:



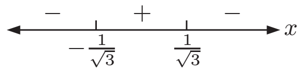
Now $f(0) = 1 - \frac{1}{1} = 1 - 1 = 0$

$\therefore$ there is a local minimum at $(0, 0)$.

ii
$$\begin{aligned} f''(x) &= \frac{2(1+x^2)^2 - 2x(2)(1+x^2)^1(2x)}{(1+x^2)^4} \\ &= \frac{2(1+x^2) - 8x^2}{(1+x^2)^3}\end{aligned}$$

$$\begin{aligned}\therefore \ f''(x) &= \frac{2-6x^2}{(1+x^2)^3} \\ &= \frac{2(1-3x^2)}{(1+x^2)^3} \\ &= \frac{2(1+\sqrt{3}x)(1-\sqrt{3}x)}{(1+x^2)^3}\end{aligned}$$

$f''(x)$ has sign diagram:



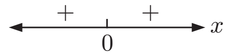
Now $f\left(-\frac{1}{\sqrt{3}}\right) = 1 - \dfrac{1}{1+\frac{1}{3}} = \frac{1}{4}$

and $f\left(\frac{1}{\sqrt{3}}\right) = 1 - \dfrac{1}{1+\frac{1}{3}} = \frac{1}{4}$

$\therefore$ there are points of inflection at $(-\frac{1}{\sqrt{3}}, \frac{1}{4})$ and $(\frac{1}{\sqrt{3}}, \frac{1}{4})$.

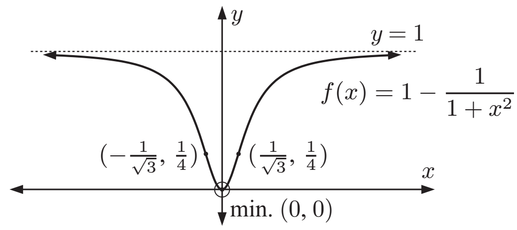
iii $f(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2}{1+x^2} - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2}$

$f(x)$ has sign diagram:

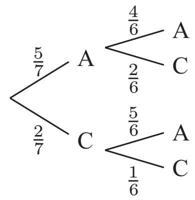


$\therefore \ f(x) \geqslant 0$ for all x.

iv



17 **a**



P(both same author)

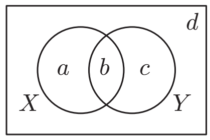
$=$ P(AA or CC)

$= \frac{5}{7} \times \frac{4}{6} + \frac{2}{7} \times \frac{1}{6}$

$= \frac{22}{42}$

$= \frac{11}{21}$

b



We are given

$$\begin{cases} a+b+c+d = 100 \\ a+b = 60 \\ b+c = 70 \\ b = 42 \end{cases}$$

$\therefore \ c = 28, \ a = 18$ and $d = 12$

i **(1)** $n(X \cup Y) = a + b + c = 88$

(2) $\text{P}(X') = \dfrac{28+12}{100} = 0.4$

(3) $\text{P}(Y \mid X') = \dfrac{c}{c+d} = \frac{28}{40} = 0.7$

ii $\text{P}(X \cap Y) = \dfrac{b}{a+b+c+d} = \dfrac{42}{100} = 0.42$

$\text{P}(X)\,\text{P}(Y) = \frac{60}{100} \times \frac{70}{100} = 0.42$

Since $\text{P}(X \cap Y) = \text{P}(X)\,\text{P}(Y)$, X and Y are independent.

18 **a** **i** $y = 10 - 3x$ cuts the x-axis when $y = 0$

$\therefore \ 10 - 3x = 0$

$\therefore \ x = 3\frac{1}{3}$

$\therefore$ P is at $(3\frac{1}{3}, 0)$.

$y = 10 - 3x$ cuts the y-axis when $x = 0$, $y = 10$

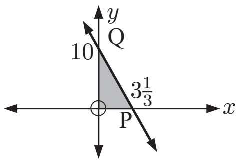
$\therefore$ Q is at $(0, 10)$.

ii Area $= \frac{1}{2} \times \text{base} \times \text{altitude}$

$= \frac{1}{2} \times 3\frac{1}{3} \times 10$

$= \frac{1}{2} \times \frac{10}{3} \times 10$

$= \frac{50}{3}$ units^2



b The gradient of [AB] is $\dfrac{2 - -4}{6 - -2} = \frac{6}{8} = \frac{3}{4}$

$\therefore$ the gradient of the perpendicular is $-\frac{4}{3}$.

M is $\left(\dfrac{-2+6}{2}, \dfrac{-4+2}{2}\right)$ or $(2, -1)$.

So, the perpendicular has equation

$$y - -1 = -\tfrac{4}{3}(x-2)$$
$$\therefore \quad 3(y+1) = -4(x-2)$$
$$\therefore \quad 3y + 3 = -4x + 8$$
$$\therefore \quad 4x + 3y = 5$$

c Area $= \displaystyle\int_1^2 \left(x + \frac{2}{x-3}\right) dx$

$$= \int_1^2 \left(x - \frac{2}{3-x}\right) dx$$
$$= \left[\frac{x^2}{2} + 2\ln(3-x)\right]_1^2 \qquad \left\{\begin{array}{l}\text{since } 3-x>0 \\ \text{on } 1 \leqslant x \leqslant 2\end{array}\right\}$$
$$= (2 + 2\ln 1) - (\tfrac{1}{2} + 2\ln 2)$$
$$= 2 - \tfrac{1}{2} - 2\ln 2$$
$$= \tfrac{3}{2} - 2\ln 2 \text{ units}^2 \qquad (a = \tfrac{3}{2},\ b = -2)$$

19 a i $\mathbf{v} + 2\mathbf{w} = 3\mathbf{i} - 7\mathbf{j} + 2(5\mathbf{i} + 2\mathbf{j})$

$= 3\mathbf{i} - 7\mathbf{j} + 10\mathbf{i} + 4\mathbf{j}$

$= 13\mathbf{i} - 3\mathbf{j}$

ii $\mathbf{v} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{9+49}$

$= \sqrt{58}$ units

$\therefore$ a unit vector is $\frac{1}{\sqrt{58}}\begin{pmatrix} 7 \\ 3 \end{pmatrix}$. $\left\{\begin{pmatrix} 3 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 0\right\}$

iii $(\mathbf{v} + \mathbf{w}) \bullet (\mathbf{v} - \mathbf{w})$

$= \mathbf{v} \bullet \mathbf{v} + \mathbf{w} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{w} - \mathbf{w} \bullet \mathbf{w}$

$= |\mathbf{v}|^2 - |\mathbf{w}|^2 \quad \{\mathbf{x} \bullet \mathbf{y} = \mathbf{y} \bullet \mathbf{x}$ and $\mathbf{x} \bullet \mathbf{x} = |\mathbf{x}|^2\}$

$= (\sqrt{58})^2 - \left(\sqrt{25+4}\right)^2$

$= 58 - 29$

$= 29$

iv $|3\mathbf{w}| = |3|\,|\mathbf{w}| = 3\sqrt{29}$ units

b i L_1 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}.$

L_2 has direction vector $\begin{pmatrix} 10 - -2 \\ -4 - 2 \\ -2 - 16 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ -18 \end{pmatrix}$

$\therefore$ L_2 is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ -2 \end{pmatrix} + s\begin{pmatrix} 12 \\ -6 \\ -18 \end{pmatrix}, \quad s \in \mathbb{R}.$

ii L_1 meets L_2 where

$$\begin{pmatrix} 4 \\ -11 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ -2 \end{pmatrix} + s\begin{pmatrix} 12 \\ -6 \\ -18 \end{pmatrix}$$

$$\therefore \quad t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - s\begin{pmatrix} 12 \\ -6 \\ -18 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ -7 \end{pmatrix}$$

$\therefore$ $t - 12s = 6$, $\quad 2t + 6s = 7$, $\quad -t + 18s = -7$

(1) (2) (3)

Adding (1) and (3), $6s = -1$ and so $s = -\frac{1}{6}$

$\therefore$ using (1), $t = 12s + 6 = -2 + 6 = 4$

In (2), $2t + 6s = 8 + (-1) = 7$ ✓

So, the lines meet at

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ 5 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}$$

which is the point $(8, -3, 1)$.

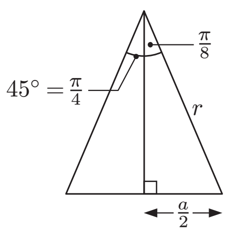
iii L_1 has direction vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

L_2 has direction vector $\begin{pmatrix} 12 \\ -6 \\ -18 \end{pmatrix} = 6\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$.

If θ is the acute angle between L_1 and L_2, then

$$\cos\theta = \frac{|\mathbf{l}_1 \bullet \mathbf{l}_2|}{|\mathbf{l}_1|\,|\mathbf{l}_2|}$$
$$= \frac{|2 - 2 + 3|}{\sqrt{1+4+1}\sqrt{4+1+9}}$$
$$= \frac{3}{\sqrt{6}\sqrt{14}}$$
$$= \frac{3}{\sqrt{84}}$$

20 a



$$\sin\tfrac{\pi}{8} = \frac{\frac{a}{2}}{r}$$
$$\therefore \quad \frac{a}{2} = r\sin\tfrac{\pi}{8}$$
$$\therefore \quad a = 2r\sin\tfrac{\pi}{8}$$

b Area of octagon $= 8 \times$ area of one triangle

$= 8 \times \frac{1}{2}r^2\sin\frac{\pi}{4}$

$= 4r^2(2\sin\frac{\pi}{8}\cos\frac{\pi}{8})$

$= 8r^2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

c i $A =$ area of base $+$ area of sides

$= 8r^2\sin\frac{\pi}{8}\cos\frac{\pi}{8} + 8 \times a \times h$

$= 8r^2\sin\frac{\pi}{8}\cos\frac{\pi}{8} + 8(2r\sin\frac{\pi}{8})h$

$\therefore$ $A = 8r(r\cos\frac{\pi}{8} + 2h)\sin\frac{\pi}{8}$ .... (1)

and $V =$ area of base $\times$ height

$\therefore$ $V = 8r^2h\sin\frac{\pi}{8}\cos\frac{\pi}{8}$ .... (2)

ii From (2), $h = \dfrac{V}{8r^2\sin\frac{\pi}{8}\cos\frac{\pi}{8}}$

Substituting into (1),

$$A = 8r\sin\tfrac{\pi}{8}\left(r\cos\tfrac{\pi}{8} + \frac{V}{4r^2\sin\frac{\pi}{8}\cos\frac{\pi}{8}}\right)$$
$$\therefore \quad A = 8r^2\sin\tfrac{\pi}{8}\cos\tfrac{\pi}{8} + \frac{2V}{r\cos\frac{\pi}{8}}$$

iii

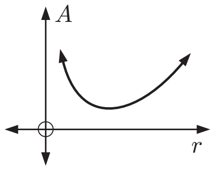
$$A = 8r^2\sin\tfrac{\pi}{8}\cos\tfrac{\pi}{8} + \frac{2V}{\cos\frac{\pi}{8}}r^{-1}$$
$$\therefore \quad \frac{dA}{dr} = 16r\sin\tfrac{\pi}{8}\cos\tfrac{\pi}{8} - \frac{2V}{\cos\frac{\pi}{8}}r^{-2}$$
$$\therefore \quad \frac{dA}{dr} = 0 \text{ when } 16r\sin\tfrac{\pi}{8}\cos\tfrac{\pi}{8} = \frac{2V}{r^2\cos\frac{\pi}{8}}$$
$$\therefore \quad r^3 = \frac{2V}{16\sin\frac{\pi}{8}\cos^2\frac{\pi}{8}}$$
$$\therefore \quad r = \left(\frac{V}{8\sin\frac{\pi}{8}\cos^2\frac{\pi}{8}}\right)^{\frac{1}{3}}$$

Now $\dfrac{d^2A}{dr^2} = 16\sin\frac{\pi}{8}\cos\frac{\pi}{8} + \dfrac{4V}{\cos\frac{\pi}{8}}\,r^{-3}$

$= 16\sin\frac{\pi}{8}\cos\frac{\pi}{8} + \dfrac{4V}{r^3\cos\frac{\pi}{8}}$

$\therefore\ \dfrac{d^2A}{dr^2} > 0$ for all $r > 0$

$\therefore$ the graph has shape



$\therefore$ the minimum value of A occurs when

$r = \left(\dfrac{V}{8\sin\frac{\pi}{8}\cos^2\frac{\pi}{8}}\right)^{\frac{1}{3}}$.

SOLUTIONS TO EXAMINATION PRACTICE SET 2

1 **a** $u_1 = 8$ and $u_{30} = u_1 + 29d = 124$

$\therefore\ 8 + 29d = 124$

$\therefore\ 29d = 116$

$\therefore\ d = \frac{116}{29} = 4$

So, $u_{12} = u_1 + 11d$

$= 8 + 11 \times 4$

$= 52$

b $S_n = \dfrac{n}{2}(u_1 + u_n)$

$\therefore\ S_{12} = \frac{12}{2}(u_1 + u_{12})$

$= 6(8 + 52)$

$= 360$

2 **a** $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 & 7 & 6 \\ -1 & 3 & 2 \\ 4 & -2 & 0 \end{pmatrix} - \begin{pmatrix} -3 & 1 & 2 \\ 2 & 1 & -5 \\ -2 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 5 & 6 & 4 \\ -3 & 2 & 7 \\ 6 & -6 & -3 \end{pmatrix}$

b $\mathbf{AB} = \begin{pmatrix} 2 & 7 & 6 \\ -1 & 3 & 2 \\ 4 & -2 & 0 \end{pmatrix}\begin{pmatrix} -3 & 1 & 2 \\ 2 & 1 & -5 \\ -2 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} -4 & 33 & -13 \\ 5 & 10 & -11 \\ -16 & 2 & 18 \end{pmatrix}$

c $|\mathbf{A}| = 2\begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} + 6\begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix}$

$= 2(4) - 7(-8) + 6(-10)$

$= 4$

d $\mathbf{B}^{-1} = \frac{1}{45}\begin{pmatrix} -23 & -5 & 7 \\ -4 & 5 & 11 \\ -10 & -10 & 5 \end{pmatrix}$ {using technology}

3 **a** $A = 180^o - 43^o - 18^o = 119^o$

Using the sine rule, $\dfrac{a}{\sin 119^o} = \dfrac{35.6}{\sin 43^o}$

$\therefore\ a = \dfrac{35.6 \times \sin 119^o}{\sin 43^o}$

$\therefore\ a \approx 45.7$

So, the distance is about 45.7 cm.

b Area $= \frac{1}{2}ab\sin C$

$\approx \frac{1}{2} \times 45.6547 \times 35.6 \times \sin 18^o$

$\approx 251\ \text{cm}^2$

4 $T_{r+1} = \binom{10}{r}(2x)^{10-r}5^r$

$= \binom{10}{r}2^{10-r}x^{10-r}5^r,\quad r = 0, 1, 2, ..., 10$

The power of x is 6 when $10 - r = 6$

$\therefore\ r = 4$

When $r = 4$, $T_5 = \binom{10}{4}2^6 5^4 x^6$

So, the coefficient of x^6 is $\binom{10}{4}2^6 5^4$ or 8 400 000.

5 **a** $\begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ {using technology}

$\therefore\ x = 3,\ y = 2$

b **i** If an inverse exists then $\begin{vmatrix} 1 & p-2 \\ 2p & p^2 \end{vmatrix} \neq 0$

$\therefore\ p^2 - 2p^2 + 4p \neq 0$

$\therefore\ 4p - p^2 \neq 0$

$\therefore\ p(4-p) \neq 0$

$\therefore\ p \neq 0$ or 4

ii The inverse is $\dfrac{1}{4p - p^2}\begin{pmatrix} p^2 & 2-p \\ -2p & 1 \end{pmatrix}$

$= \begin{pmatrix} \dfrac{p^2}{4p-p^2} & \dfrac{2-p}{4p-p^2} \\ \dfrac{-2p}{4p-p^2} & \dfrac{1}{4p-p^2} \end{pmatrix}$

6 $\mu = 3.18$, $\sigma = 0.195$, and let X be the height of a randomly selected tree.

a **i** $P(X > 3) \approx 0.822$

ii $P(2.8 < X < 3.3) \approx 0.705$

b We need to find k such that $P(X \geqslant k) = 0.2$ or $P(X \leqslant k) = 0.8$.

$\therefore\ P\left(\dfrac{X - 3.18}{0.195} \leqslant \dfrac{k - 3.18}{0.195}\right) = 0.8$

$\therefore\ P\left(Z \leqslant \dfrac{k - 3.18}{0.195}\right) = 0.8$

$\therefore\ \dfrac{k - 3.18}{0.195} = 0.841\,621$

$\therefore\ k - 3.18 = 0.164\,116$

$\therefore\ k \approx 3.344$

So, a tree must be about 3.34 m high to be in the tallest 20%.

7 **a** $f(x) = 3\sin(2x + 1)$

i $f'(x) = 3\cos(2x + 1) \times 2$

$= 6\cos(2x + 1)$

ii $\int f(x)\,dx = \int 3\sin(2x + 1)\,dx$

$= 3(\frac{1}{2})(-\cos(2x + 1)) + c$

$= -\frac{3}{2}\cos(2x + 1) + c$

b $\int e^x(1 - e^{2x})\,dx = \int(e^x - e^{3x})\,dx$

$= e^x - \frac{1}{3}e^{3x} + c$