TOPIC 1: ALGEBRA

SEQUENCES AND SERIES

A **number sequence** is a set of numbers defined by a rule which is valid for positive integers. Often, the rule is a formula for the **general term** or **nth term** of the sequence.

Arithmetic Sequences

In an **arithmetic sequence**, each term differs from the previous one by the same fixed number.

$u_{n+1} - u_n = d$ for all n, where d is a constant called the **common difference**.

For an arithmetic sequence with first term u_1 and common difference d, $u_n = u_1 + (n-1)\,d$.

Geometric Sequences

In a **geometric sequence**, each term is obtained from the previous one by multiplying by the same non-zero constant.

$\frac{u_{n+1}}{u_n} = r$ for all n, where r is a constant called the **common ratio**.

For a geometric sequence with first term u_1 and common ratio r, $u_n = u_1 r^{n-1}$.

Series

A **series** is the addition of the terms of a sequence. Given a series which includes the first n terms of a sequence, its sum is $S_n = u_1 + u_2 + ... + u_n$.

Using **sigma notation** we can write

$u_1 + u_2 + u_3 + ... + u_n$ as $\sum_{k=1}^{n} u_k$.

For an **arithmetic series**, $S_n = \frac{n}{2}(u_1 + u_n)$.

For a **geometric series**, $S_n = \frac{u_1(r^n - 1)}{r - 1}$.

The sum of an **infinite geometric series** is

$S = \frac{u_1}{1-r}$ provided $|r| < 1$.

If $|r| > 1$ the series is **divergent**.

For compound interest problems we have a geometric sequence. If the interest rate is $i\%$ per time period then the common ratio is $\left(1 + \frac{i}{100}\right)$ and the number of compounding periods is n.

EXPONENTIALS AND LOGARITHMS

Exponential and logarithmic functions are inverses of each other. The graph of $y = \log_a x$ is the reflection in the line $y = x$ of the graph of $y = a^x$.

Index or Exponent Laws	
$a^x \times a^y = a^{x+y}$	$a^{-x} = \frac{1}{a^x}$ and $\frac{1}{a^{-x}} = a^x$
$\frac{a^x}{a^y} = a^{x-y}$	$a^{\frac{1}{n}} = \sqrt[n]{a}$
$(a^x)^y = a^{xy}$	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$
$a^0 = 1 \ (a \neq 0)$	

If $a^x = a^k$ then $x = k$. So, if the base numbers are the same, we can **equate indices**.

If $b = a^x$, $a \neq 1$, $a > 0$, we say that x is the **logarithm** of b in base a, and that $b = a^x \Leftrightarrow x = \log_a b$, $b > 0$.

The **natural logarithm** is the logarithm in base e. $\ln x \equiv \log_e x$

Logarithm Laws	
Base a	**Base e**
$\log_a xy = \log_a x + \log_a y$	$\ln xy = \ln x + \ln y$
$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\ln \left(\frac{x}{y}\right) = \ln x - \ln y$
$\log_a x^y = y \log_a x$	$\ln x^y = y \ln x$
$\log_a 1 = 0$	$\ln 1 = 0$
$\log_a a = 1$	$\ln e = 1$

To change the base of a logarithm, use the rule $\log_b x = \frac{\log_c x}{\log_c b}$.

$x = \log_a a^x$ and $x = a^{\log_a x}$ provided $x > 0$.

THE BINOMIAL THEOREM

$a + b$ is called a **binomial** as it contains two terms.

Any expression of the form $(a+b)^n$ is called a **power of a binomial**.

The **general binomial expansion** is

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + ... + \binom{n}{r} a^{n-r}b^r + ... + \binom{n}{n} b^n$$

where $\binom{n}{r}$ is the binomial coefficient of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, ..., n$.

The **general term** in the binomial expansion is

$T_{r+1} = \binom{n}{r} a^{n-r}b^r$, so $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$.

You should know how to calculate binomial coefficients from Pascal's triangle and using your calculator.

SKILL BUILDER QUESTIONS (NO CALCULATORS)

1 Find the 20th term of:

a 51, 45, 39, 33, ... **b** 0.125, 0.5, 2, 8, ...

2 Find the sum of the infinite geometric series with first term 27 and fourth term 8.

3 The 7th and 15th terms of an arithmetic sequence are 1 and -23 respectively. Find:

a the 27th term

b the sum of the first 27 terms of the sequence.

4 Find the 10th term of:

a $x^2, 3x^2, 5x^2, ...$ **b** $x^{-\frac{1}{2}}, x, x^{2\frac{1}{2}}, ...$

5 The first term of a finite arithmetic series is 18 and the sum of the series is -210. If the common difference is -3, find the number of terms in the series.

6 Simplify:

a $(5x^2)^3 \times \left(\frac{x}{4}\right)^2 \times \frac{8}{25x^5}$ **b** $\frac{24a^3b^8}{15(a^2b)^3} \div \frac{5ab^3}{12a^6b}$

c $\frac{8x^{-2}y^3}{3(xy^2)^0} \times \frac{9x^0y^{-1}}{4x^{-3}}$ **d** $\frac{4x^{\frac{1}{2}} \times x^{-1\frac{1}{2}}}{8x^2}$

e $\sqrt[5]{a^3} \times \sqrt{a^5}$

7 Find:

a $16^{-\frac{1}{2}}$ **b** $81^{\frac{1}{4}}$ **c** $32^{\frac{2}{5}}$

d $27^{-\frac{4}{3}}$ **e** $\left(\frac{1}{9}\right)^{\frac{3}{2}}$ **f** $(0.008)^{-\frac{5}{3}}$

8 Factorise:

a $9^x - 6(3^x) + 8$ **b** $25^x + 5^{x+1} + 6$

9 Write in the form $\log A$:

a $\log 2 + \log 12$ **b** $\log 36 - \log 12$

c $3\log 5 + 2\log 3$ **d** $\frac{1}{4}\log 81$

10 Find x if $8^{2x-3} = 16^{2-x}$.

11 Simplify: $\dfrac{3^{x+1} - 3^x}{2(3^x) - 3^{x-1}}$

12 Solve for x: $4^x + 4 = 17(2^{x-1})$

13 If $A = \log_{10} P$, $B = \log_{10} Q$ and $C = \log_{10} R$, express in terms of A, B and C:

a $\log_{10}\left(P^2 Q\sqrt{R}\right)$ **b** $\log\left(\dfrac{R^3}{(PQ^2)^4}\right)$

14 If $\log_5(2x - 1) = -1$, find x.

15 Write $\dfrac{8}{\log_5 9}$ in the form $a\log_3 b$ where $a, b \in \mathbb{Z}$.

16 Write $2\ln x + \ln(x-1) - \ln(x-2)$ as a single logarithm.

17 Solve for x: $\log_3 x + \log_3(x-2) = 1$.

18 If $\log_a 5 = x$, find in terms of x:

a $\log_a(5a)$ **b** $\log_a\left(\dfrac{a^2}{25}\right)$.

19 Write as a logarithmic equation in base b:

a $M = ab^3$ **b** $D = \dfrac{a}{b^2}$

20 Write without logarithms:

a $\log_{10} M = 2x - 1$ **b** $\log_a N = 2\log_a d - \log_a c$

21 Use Pascal's triangle to help expand and simplify $\left(x + \dfrac{1}{x}\right)^5$.

SKILL BUILDER QUESTIONS (CALCULATORS)

1 Find the first term in the arithmetic sequence 100, 130, 160, 190, ... to exceed 1200.

2 If the 5th and 8th terms of a geometric sequence are 18 and 486 respectively, find the 12th term.

3 Maria invested €800 on Jan 1st 1996.

a If her investment earns interest of 7% per annum, what was it worth on Jan 1st 2004?

b How many years will it take for her investment to reach €4000?

4 Ying is training to run a marathon. In one week she ran 10 km on the first day and increased the distance by 10% on each subsequent day.

a How far did she run on the seventh day?

b What was the total distance she ran during the week?

5 Find the sum of the series:

a $10 + 14 + 18 + 22 + \ldots + 138$

b $6 - 12 + 24 - 48 + 96 - \ldots + 1536$

6 The sum of an infinite geometric series is 1.5, and its first term is 1. Find:

a the common ratio

b the sum of its first 7 terms in rational form.

7 A sequence is defined by $u_n = 12\left(\frac{2}{3}\right)^{n-1}$.

a Prove that the sequence is geometric.

b Find the 5th term in rational form.

c Find: **i** $\sum_{n=1}^{\infty} u_n$

ii $\sum_{n=1}^{20} u_n$ correct to 4 decimal places.

8 Stan invests £3500 for 33 months at an interest rate of 8% p.a. compounded quarterly. Find its maturing value.

9 Solve for x:

a $3^x = 243$ **b** $5 \times 2^x = 160$ **c** $e^x = 27$

d $(1.25)^x = 10$ **e** $7e^x = 100$

10 Solve for t:

a $200 \times e^{\frac{t}{4}} = 1500$ **b** $0.15e^{0.012t} = 2.18$

c $20e^{0.2t} = 250$

11 Solve for x: $x^2 > e^{-x}$.

12 Find the constant term in the expansion of $\left(x + \dfrac{3}{x^2}\right)^9$.

13 Find the coefficient of x^4y^9 in the expansion of $\left(x + 2y^3\right)^7$.

14 Consider the binomial expansion of $\left(2x - \dfrac{1}{x^2}\right)^{12}$. Find:

a the coefficient of x^3 **b** the constant term.

15 Find k given that the constant term of $\left(kx + \dfrac{1}{\sqrt{x}}\right)^9$ is $-10\frac{1}{2}$.

16 Find the coefficient of x^5 in the expansion of $(x+2)(1-x)^{10}$.

17 If $A = 125e^{-kt}$ and $A = 200$ when $t = 3$, find the value of k.

18 Twins Pierre and Francesca were each given 100 dollars on their 15th birthday. They immediately put their money into their individual money boxes.

Each week throughout the next year they added a portion of their weekly pocket money. Pierre added $10 each week. Francesca added 50 cents the first week, $1 the next, $1.50 the next, and so on, adding an extra 50 cents each subsequent week.

a Find the amount that each had added to his or her money box after 8 weeks.

b How much did Francesca add to her money box on her 16th birthday?

c Calculate the total amount they each had in their money boxes after one year.

19 Hayley and Patrick are training for a road cycling race. During the first week Hayley cycles 60 km and Patrick does likewise. Hayley cycles an additional 20 km each subsequent week, whereas Patrick increases his distance by 20% each subsequent week.

a How far does each of them cycle in the 5th week of training?

b Who is the first to cycle 210 km in one week?

c What total distance does each cycle in the first 12 weeks?

20 Kapil invested 2000 rupees in a bank account on Jan 1st 2002. Each year thereafter, he invested another 2000 rupees into the same account which pays 8.25% per annum compounded annually.

a Find the total value of his investment immediately after he invested 2000 rupees on Jan 1st 2009.

b Would it have been a better option for Kapil to invest his money each year into an account paying 9% per annum simple interest? Justify your answer.

21 At the beginning of 2000 the number of koalas on an island was 2400. The numbers steadily increase each year according to $N(t) = 2400 + 250t$ where t is the number of years since 2000. The population of kangaroos on the same island is given by $K(t) = 3200 \times (0.85)^t$.

a What was the kangaroo population at the beginning of 2000?

b How many kangaroos were on the island at the beginning of 2005?

c How many koalas were on the island at the beginning of 2005?

d After how many years will the kangaroo population fall below 1000?

e When will the number of koalas exceed the number of kangaroos?

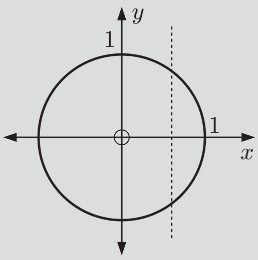
TOPIC 2: FUNCTIONS AND EQUATIONS

FUNCTIONS $f : x \mapsto f(x)$ OR $y = f(x)$

A **relation** is any set of points on the Cartesian plane.

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first member. For each value of x there is only one value of y or $f(x)$. We sometimes refer to y or $f(x)$ as the **image value** of x.

We **test for functions** using the vertical line test. A graph is a function if no vertical line intersects the graph more than once.



For example, a circle such as $x^2 + y^2 = 1$ has a graph which is not a function.

The **domain** of a function is the set of values that x can take.

To find the domain of a function, remember that we cannot:

- divide by zero
- take the square root of a negative number
- take the logarithm of a non-positive number.

The **range** of a function is the set of values that y or $f(x)$ can take.

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g is $f \circ g : x \mapsto f(g(x))$.

In general, $f(g(x)) \neq g(f(x))$, so $f \circ g \neq g \circ f$.

The **identity function** is $f(x) = x$.

If $y = f(x)$ has an **inverse function** $y = f^{-1}(x)$, then the inverse function:

- must satisfy the vertical line test
- is a reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- has range equal to the domain of $f(x)$
- has domain equal to the range of $f(x)$.

The **reciprocal function** is $f(x) = \dfrac{1}{x}$, $x \neq 0$.

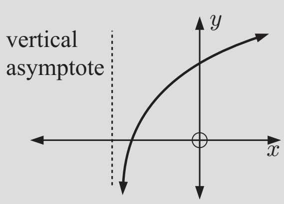
The reciprocal function is a **self-inverse function**, as $f^{-1}(x) = f(x) = \dfrac{1}{x}$.

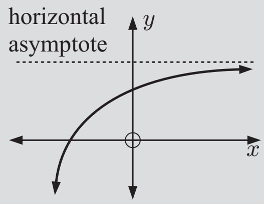
GRAPHS OF FUNCTIONS

The **x-intercepts** of a function are the values of x for which $y = 0$. They are the **zeros** of the function.

The **y-intercept** of a function is the value of y when $x = 0$.

An **asymptote** is a line that the graph *approaches* or begins to look like as it tends to infinity in a particular direction.





To find vertical asymptotes, look for values of x for which the function is undefined:

- if $y = \dfrac{f(x)}{g(x)}$ find where $g(x) = 0$
- if $y = \log_a(f(x))$ find where $f(x) = 0$.

To find horizontal asymptotes, consider the behaviour as $x \to \pm\infty$.

Transformations of graphs

- $y = f(x) + b$ **translates** $y = f(x)$ vertically b units.
- $y = f(x - a)$ **translates** $y = f(x)$ horizontally a units.
- $y = f(x-a)+b$ **translates** $y = f(x)$ by the vector $\binom{a}{b}$.
- $y = pf(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with dilation factor p.
- $y = f\left(\dfrac{x}{q}\right)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with dilation factor q.
- $y = -f(x)$ is a **reflection** of $y = f(x)$ in the x-axis.
- $y = f(-x)$ is a **reflection** of $y = f(x)$ in the y-axis.
- $y = f^{-1}(x)$ is a **reflection** of $y = f(x)$ in the line $y = x$.

LINEAR FUNCTIONS

A **linear function** has the form $f(x) = ax + b$, $a \neq 0$.

Its graph is a straight line with gradient a and y-intercept b.

Perpendicular lines have gradients which are the negative reciprocals of each other.

QUADRATIC FUNCTIONS

A **quadratic function** has the form $f(x) = ax^2 + bx + c$, $a \neq 0$.

The graph is a parabola with the following properties:

- it is *concave up* if $a > 0$

  and *concave down* if $a < 0$