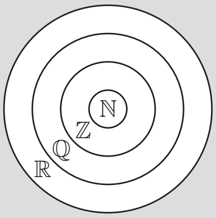
SUMMARY

NUMBER SETS

- Natural Numbers $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$
- Integers $\mathbb{Z} = \{...., -2, -1, 0, 1, 2, ...\}$
- Rational Numbers $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, \ q \neq 0\}$
- Real Numbers $\mathbb{R} = \{\text{all real numbers}\}$
  These are numbers that can be placed on a number line.



The Venn diagram shows the relationship between the sets.

You must understand and be able to use:

- ► Prime numbers, prime factors, common factors, multiples.
- ► Universal Set, subsets, complements, intersection, union.
- ► Venn diagrams and the link to Topic 3 (Sets, Logic and Probability).

Example:

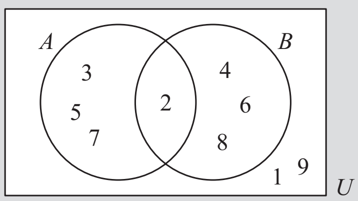
For $U = \{n \in \mathbb{Z} \mid 0 < n < 10\}$, $A = \{\text{prime numbers}\}$, and $B = \{\text{even numbers}\}$:

a List the elements of each set.
b Represent the sets A and B on a Venn diagram.
c Find the number of elements in $A \cap B$, $A \cup B$ and $(A \cup B)'$.
d Write down an element of B.
e Write down a subset of B.

Solution:

a $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$

b



c $A \cap B = \{2\}$, $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$
and $(A \cup B)' = \{1, 9\}$
$\therefore \quad n(A \cap B) = 1 \qquad n(A \cup B) = 7 \qquad n(A \cup B)' = 2$

d An element of B is 4 (or 6 or 8 or 2).

e A subset of B is $\{2, 4\}$, for example.

APPROXIMATION AND ESTIMATION

You must be familiar with and use:

- ► Rounding: e.g., 1.50 = 2 (nearest whole number)
  1.49 = 1 (nearest whole number)
- ► Significant figures:
  The significant figures are counted from the first left hand non-zero digit.
  e.g., count from the 3 in the following examples.
  345 600 = 346 000 (3 s.f.)
  and 0.003 456 = 0.00346 (3 s.f.)

PERCENTAGE ERROR

$$\text{Percentage error} = \frac{\text{approximate value} - \text{exact value}}{\text{exact value}} \times 100\%$$

Note: Some % answers work out to be negative. We convert these to the positive.

Example:

A length of rope, measured exactly as 7.45 m, is stated as 7.4 m long on the packaging. Find the percentage error caused by the rounding.

Solution:

$$\%\text{ error} = \frac{7.4 - 7.45}{7.45} \times 100 = -0.671\%$$

So, the percentage error is 0.671%.

STANDARD FORM (SCIENTIFIC NOTATION)

Standard form is $a \times 10^k$ where $1 \leqslant a < 10$ and $k \in \mathbb{Z}$.

Note that

- 2565 in standard form is 2.565×10^3
- 3.04×10^{-3} as a decimal is 0.003 04

SI UNITS

You must understand and be able to use:

- ► Metric conversion e.g., convert 5.3 g = 530 cg
- ► Areas and Volumes e.g., 8 m × 5 m = 40 m^2
- ► Formulae e.g., converting temperature units
- ► Units such as cms^{-1} (cm/s)
  These arise when doing divisions such as $\frac{\text{distance}}{\text{time}}$

  Example:

  Given speed $= \frac{\text{distance}}{\text{time}}$, find the time taken for a car travelling at an average speed of 70 kmh^{-1} to cover 245 km.

  Solution: speed $= \frac{\text{distance}}{\text{time}}$
  $\therefore \quad 70 = \frac{245}{t}$
  $\therefore \quad t = \frac{245}{70}$
  $\therefore \quad t = 3.5$ hours

SEQUENCES

These are sequences of numbers such as

23, 27, 31, 35, 39, ... or 24, 12, 6, 3, 1.5, ... say

In general, we write: $u_1, u_2, u_3, u_4, u_5, ...$

Special sequences are:

- **Arithmetic**
  with common difference, $d = u_2 - u_1 = u_3 - u_2 = ...$
  and $\boldsymbol{u_n = u_1 + (n-1)d}$
- **Geometric**
  with common ratio $r = \frac{u_2}{u_1} = \frac{u_3}{u_2} = ...$
  and $\boldsymbol{u_n = u_1 r^{n-1}}$

Example 1:

Find the common difference or common ratio in these sequences:

a 4, 13, 22, 31, ... **b** $\frac{1}{2}$, 1, 2, 4, ...

Solution: **a** This is arithmetic as
$13 - 4 = 22 - 13 = 31 - 22 = 9 \quad \therefore \ d = 9$

b This is geometric as $\frac{4}{2} = \frac{2}{1} = \frac{1}{\frac{1}{2}} = 2 \quad \therefore \ r = 2$

Example 2:

Which term of the sequence 3, 7, 11, ... is 119?

Solution:

The sequence is arithmetic with $u_1 = 3$, $d = 4$,

Now $u_n = u_1 + (n-1)d$

$\therefore \quad 119 = 3 + (n-1) \times 4$

$\therefore \quad n = 30$ {using a gdc}

SERIES

A series is the addition of the terms of a sequence.

That is $u_1 + u_2 + u_3 + u_4 + ...$

These formulae can be used to find the **sum** of a series:

- **Arithmetic:** $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ or

  $S_n = \frac{n}{2}(u_1 + u_n)$

- **Geometric:** $S_n = \frac{u_1(r^n - 1)}{r - 1}$ or

  $S_n = \frac{u_1(1 - r^n)}{1 - r}$

APPLICATIONS FOR SEQUENCES AND SERIES

Applications involve:

- ► Simple and Compound Interest
- ► Growth and Decay

Example:

A 50 g sample of radioactive material decreases in weight by 3.2% per annum. Calculate the number of years it will take for the sample to weigh less than 20 g.

Solution:

The decay is geometric with

$u_1 = 50$, $u_n = 20$ and $r = 96.8\% = 0.968$

Using $u_n = u_1 \times r^{n-1}$ we have the equation

$20 = 50 \times (0.968)^{n-1}$

$\therefore \quad n = 29.173 = 30$ years {using a gdc}

SIMULTANEOUS EQUATIONS

An example is: $2a + b = 7$

$3a + 2b = 11$

These may be solved using

- Substitution
- Elimination
- a gdc

  (The use of the gdc is emphasised in the syllabus.)

The solution is $a = 3$, $b = 1$.

QUADRATIC EQUATIONS

These are equations of the form $ax^2 + bx + c = 0$.

You must understand and be able to use:

- ► methods to solve equations of the form $ax^2 + bx + c = 0$
- ► terms such as factors, zeros and roots

  Example: $6x^2 - x - 2 = 0$ is a quadratic equation

  $(2x+1)(3x-2) = 0$ is its factored form

  $x = -\frac{1}{2}$ are $\frac{2}{3}$ are its roots or solutions
- ► a gdc to solve quadratic equations

APPLICATIONS FOR SIMULTANEOUS AND QUADRATIC EQUATIONS

You must understand and be able to:

- ► write equations in terms of x
- ► use algebra to solve problems

TOPIC 2 – NUMBER AND ALGEBRA (SHORT QUESTIONS)

1 **a** List the elements of the following sets:

i $A = \{x \in \mathbb{Z} \mid -2 < x < 3\}$

ii $B = \{$prime numbers less than 15$\}$

iii $C = \{x \in \mathbb{R} \mid x^2 = 8\}$

b State whether the following statements are true or false.

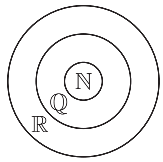
i All rational numbers are integers.

ii $4 - p > 4 + p, \quad p \in \mathbb{Z}^-$.

iii $\mathbb{N} = \{0, 1, 2, 3, ...\}$

2 Place the following numbers in the appropriate Venn diagram.

2.5, π, -3, $\frac{8}{5}$, 0, $\sqrt{25}$



3 $U = \{$natural numbers less than 12$\}$ $A = \{$multiples of 3$\}$

$B = \{$factors of 10$\}$

List the elements of: **a** $A \cap B$ **b** $A \cup B$ **c** $(A \cup B)'$.

4 **a** Draw a number line to represent the set $X = \{x \in \mathbb{R} \mid -3 \leqslant x \leqslant 3\}$

b Use a number line to clearly represent each of the following:

i $x < -1$ **ii** $0 \leqslant x < 2$ **iii** $x \geqslant 2$

5 $U = \{n \in \mathbb{N} \mid 1 \leqslant n \leqslant 12\}$

$P = \{$odd numbers$\}$, $Q = \{$factors of 12$\}$ and

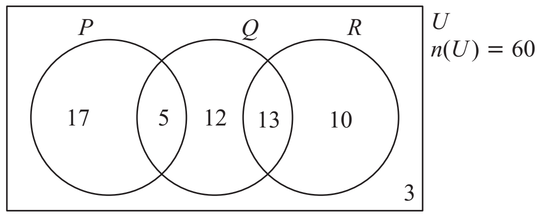
$R = \{$multiples of 5$\}$

List the elements of

a $P \cap Q$ **b** $(P \cap Q) \cup R$

c $Q \cap R$ **d** $P' \cap (Q \cup R)$

6 The information below shows the number of different types of movies available for hire. $P = \{$comedy$\}$, $Q = \{$romance$\}$, $R = \{$adventure$\}$



Find

a $n(P)$ **b** $n(P \cup Q)$ **c** $n(Q \cap R)$

d $n((P \cup Q \cup R)')$ **e** $n(Q')$ **f** $n(P \cap R)$

7 **a** A number of IB students study English or Spanish or both English and Spanish. 25 study Spanish and 18 study English. If 6 students study both languages, how many IB students are there?

b An international school offers its programme in both French and English languages. 60% of the students study in the English language programme and 76% study in the French language programme. What percentage of the students take lessons in both languages?

c In a class of 25 students, 20% do not study art or drama. 13 students study art and 9 students study drama. How many students study both art and drama?

8 Evaluate $\sqrt{\dfrac{32.76}{3.95 \times 2.63}}$, giving your answer:

a correct to 3 decimal places
b correct to the nearest whole number
c correct to 3 significant figures
d in standard form.

9 The speed of light is approximately 186 280 miles per second. Assume the distance from Mars to Earth is approximately 195 000 000 km.

a Given that mile: kilometre $= 1 : 1.609$, determine the speed of light in kilometres per minute. Give your answer to 3 significant figures.
b Express your answer to **a** using scientific notation (standard form).
c If a light source on Mars is ignited, how many minutes will it be before it is seen through a telescope on Earth?

10 **a** A measurement of 5.645 cm is rounded, correct to 3 significant figures.

i Write down the actual error caused by rounding.
ii Calculate the relative error.
iii Calculate the percentage error.

b The speedometer of a car has an error of 3.2% at 70 km per hour.

i What is the actual error?
ii Write down the extreme possible values for the true speed of the car.

11 The length of a section of pipe is stated as 4 m. Claudia carefully measures the pipe and finds the actual length to be 3.94 m.

a Write down the size of the error in the stated length.
b Five sections of pipe are joined together.
Find the actual length of the joined pipes.
c Write down the error for the joined pipes if the actual length of each pipe is rounded to the nearest metre.
d Calculate the percentage error of the stated length against the actual length of joined pipes.

12 The first three terms of a sequence are -2, -9, and -16.

a Write down the next two terms of the sequence.
b Draw a mapping diagram of the first 5 terms.
c Find a formula for the nth term of the sequence.

13 **a** Write down the first 3 terms of the sequence given by:

$$u_n = n(n+1)$$

b Find the 15th term.
c Which term of this sequence is 600?

14 The first three terms of an arithmetic sequence are -347, $k - 166$ and -185.

a Find the value of k.
b Find a formula for the nth term of the sequence.
c Which is the first positive term of the sequence?

15 The sixth term of an arithmetic sequence is 49 and the fifteenth term is 130.

a Find the common difference for this sequence.
b Find the first term.
c How many of the terms of this sequence have a value which is less than 300?

16 The sum of the first 7 terms of an arithmetic series is 329. The common difference is 14.

a Find the value of the first term.
b 69 800 is the sum of the first n terms of the sequence. Find n.

17 The first three terms of a geometric sequence are 0.75, 2.25 and 6.75.

a Find the common ratio.
b Write down a formula for the nth term.
c Calculate the sum of the first 10 terms.

18 A rubber ball is dropped vertically from a height of 5 m. It bounces up to a height of 4.5 m on the first bounce, then to 4.05 m on the second bounce and so on.

a Find the common ratio.
b Calculate the height of the third bounce.
c How far has it travelled vertically by the time it strikes the floor for the fourth time?

19 128 football clubs enter the first round of a knockout competition. In each round, half the participants are eliminated.

a How many remain in the second and in the third rounds?
b If there are n rounds, how many participants remain in the nth round?
c Calculate the number of rounds needed in this competition to determine a winner.

20 The second term of a geometric sequence is 14.5 and the fifth term is 1.8125.

a Determine the common ratio.
b Find the value of the first term.
c Find the sum of the first 5 terms.

21 The population of a small town increases by an average of 9% per annum. In 2000, the population was 1200.

a Calculate the size of the population in 2005.
b In which year will the population reach 2500?
c Find the rate of increase that would result in the population reaching 3200 in 2010.

22 **a** Find the term that the sequences, $u_n = 178 - 4n$ and $u_n = 7n + 57$, have in common.

b A firm's revenue function is $R = 25n$ and its cost function is $C = 21000 + 7.5n$, where n is the number of goods produced and sold. Find the value of n for the Cost to equal the Revenue for the firm.
c The sum of the first n natural numbers is equal to $\dfrac{n(n+1)}{2}$. For what values of n does the sum exceed 435?

23 **a** Solve, using technology:

i $12s + 17r = 277$, $5s + 11r = 135$ **ii** $u_1 + 27d = 162$, $35d = 202 - u_1$

b The cost of hiring a taxi includes a flat fee of $\$a$ plus $\$p$ per kilometre. A 12 kilometre taxi ride costs \$20 and a 22 kilometre journey costs \$34. Find the values of a and p.

24 The perimeter of a rectangle is 80 cm. The width is x cm.

a Write down the value of the length, in terms of x.
b Show that the area A of the rectangle is given by the function $A = 40x - x^2$.
c The area of the rectangle is 375 cm^2. Find its length.

25 The approximate height above the ground (s) of a sky rocket at any time (t) seconds after firing is given by $s = ut - 5t^2$, where u represents the initial speed of the sky rocket. If the initial speed is 70 ms^{-1}, find:

a the amount of time the rocket is in the air.
b the time the rocket is above 30 m.

TOPIC 2 – NUMBER AND ALGEBRA (LONG QUESTIONS)

1 **a** Which of the following statements are false? Justify your answer.

i $\{-2, -1, 0, 1, 2\} \subset \{x \in \mathbb{R} \mid x < 2\}$
ii $\{0, 1, 2, 3, 4\} \subset \{x \in \mathbb{Z} \mid x \leqslant 5\}$

iii $\{x \in \mathbb{Z} \mid x^2 + x = 2\} = \{-1, 1, 2\}$

b $U = \{x \in \mathbb{N} \mid 14 \leqslant x < 30\}$

$A = \{\text{multiples of } 7\}$, $B = \{\text{factors of } 56\}$,
$C = \{\text{even numbers} \geqslant 20\}$

i List the members of each of the sets A, B and C contained in U.

ii Represent these sets of numbers on a 3-circle Venn diagram.

iii List the members of the following sets:

a $A \cap B \cap C$ **b** $(A \cap B) \cup C$
c $(A \cap B)' \cap C$ **d** $(A \cup C)' \cap B$

c p and q are different integers. Which of the following statements are false? Give an example to support your decision.

i $p + q = q + p$ **ii** $p - q = q - p$ **iii** $pq = qp$

2 a Draw a number line to represent $\{x \in R \mid -5 \leqslant x \leqslant 5\}$. Represent the following on the number line using appropriate notation.

i $x \geqslant 4$ **ii** $1 \leqslant x < 3$
iii $-3 \leqslant x \leqslant 0$ **iv** $x < -4$

b Let $X = \{x \in \mathbb{Z} \mid -4 \leqslant x \leqslant 4\}$ and
$Y = \{y \in \mathbb{Z} \mid -4 \leqslant y \leqslant 4\}$.

On a set of coordinate axes, plot the points which represent:

i the set $W = \{(x, y) \mid \frac{y}{x} = -1, \ x \in X, y \in Y\}$

ii the set
$V = \{(x, y) \mid x \in X, y \in Y, x \geqslant 0, y \geqslant 0, y - x = 1\}$

c If $x, y \in \mathbb{Q}$ and $x > y$, write down an example which makes $x^2 < y^2$ true.

3 a Evaluate $25.32 \times \dfrac{6.057}{2.4 \times \sqrt{5.14}}$, giving your answer correct to:

i five significant figures
ii the nearest tenth
iii 1 significant figure.

b Three sections of fencing are erected. Each section has a stated length of 3.60 m, measured to the nearest centimetre. The actual length of each section is 3.63 m.

i Find the actual length covered by the three sections of fencing.

ii Calculate the percentage error between the actual length and stated length of the three sections of fencing.

c The three sections of fencing of part **b** form one side of a square enclosure. The enclosure will have a concrete floor 100 ± 5 mm thick. Concrete costs €47.50 per cubic metre.

i Write down the maximum and minimum possible values for the volume of concrete needed for the floor.

ii Calculate the difference in cost between the maximum possible volume of concrete and the planned volume based on the stated length.

iii Express this cost difference as a percentage of the planned cost.

4 a Write down the values of the first three terms of each of the following:

i $u_n = 120 + 3(n - 1)$ **ii** $u_{n+1} = u_n + 7, \quad u_1 = 4$

b i Which term do the sequences listed in **a** have in common?

ii What is the value of that term?

c Which of the sequences has 151 as one of its terms?

d The sum of the first n terms of the sequences listed in **a** is the same. Find n.

e When will the difference between the sums of the sequences in **a** be 228?

5 The first 3 terms of a sequence are 56, 28 and 14.

a i Show that the sequence is geometric.

ii Find the 8th term of the sequence.

iii Find the sum of the first 8 terms.

b The third term of another geometric sequence is 24.5 and 5th term is 12.005 .

i Find the first term and the common ratio.

ii Write down the general formula for a term of this sequence.

c The first n terms of the sequence in **b** are larger than the corresponding terms for the sequence $u_n = 20 \times (0.8)^{n-1}$. Find n.

d For the sequence $u_n = 20 \times (0.8)^{n-1}$, find the sum of the first **i** 30 terms **ii** 50 terms **iii** 100 terms. Answer to 3 s.f. in each case. Comment on your results.

6 a Misha takes out a loan to purchase a generator for his business. He borrows \$20 000 at 12.5% per annum, compound interest. At the end of each year, Misha is required to pay $\$k$.

i If Misha does repay $\$k$ each year, explain why the amount owing on the loan at the end of the first year is $\$(20\,000 \times 1.125) - k$.

ii Write down an expression, in terms of k, for the amount owing on the loan at the end of the second year.

iii At the end of the second year, the amount owing on the loan will be \$17 131.25. Find k.

b Misha paid a total of \$24 000 for the generator. Its value depreciated after purchase so that, at the end of the first year, it was only worth \$20 400. The value in each subsequent year would decline at the same rate.

i Find the percentage decrease in the value of the generator in the first year.

ii Calculate the value of the generator at the end of the second year.

iii Write down the formula for the value after n years.

iv Sketch the graph of the value of the generator for the first 6 years.

7 Tahlia borrows \$5000 from her bank. Her repayments each month are \$250 plus the interest for that month. Interest is charged at 1.2% each month.

a Determine the payments she must make at the end of the:

i first month **ii** second month **iii** third month.

These monthly payments form an arithmetic sequence.

b Find the first term and common difference for the sequence and hence write down the formula to find the value of the nth payment for this loan.

c Calculate the size of the payment at the end of the tenth month.

d Determine the number of payments required for Tahlia to pay off the loan.

e Calculate the total amount Tahlia will pay for this loan.

8 A radioactive material loses 10% of its weight per year. The weight of the material at the start of the first year was 200 g.

a i Write down the weight of the material at the start of the second and third years.

ii Write down the common ratio for the geometric sequence formed by the yearly weight of the material.

iii Find the weight of the material at the start of the 6th year.

iv Draw a sketch of the annual weight of the material for the first 6 years.

v At the start of which year will the weight of the material fall below 20 g?

b The weight of a second sample of radioactive material decreased from 120 g to 49.152 g after 6 years. Find the annual percentage rate of decrease.